Periodic semi-linear functional differential inclusion system with state-dependent delays

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Abstract:

Introduction/Purpose: The study addressed the existence and uniqueness of mild solutions for a class of periodic semi-linear functional differential inclusion systems with state-dependent delays. Such systems are of growing interest due to their wide range of applications in control theory, biological systems, and engineering models involving time delays and uncertainties.

Methods: The investigation was conducted within the framework of Banach spaces. The Perov fixed point theorem, known for its effectiveness in handling systems with multiple components, was applied to establish the existence of mild solutions. The analytical approach incorporated techniques suitable for differential inclusions and operator theory, particularly focusing on systems influenced by state-dependent delays.

Results: Several results were obtained concerning the existence and uniqueness of mild solutions for the considered system. The application of the Perov fixed point theorem allowed the derivation of sufficient conditions under which such solutions exist. Furthermore, the periodic nature of the system was accounted for, ensuring the solutions adhered to the prescribed temporal structure.

Conclusions: The findings confirmed that under specific assumptions, the system admits at least one mild solution. The results contributed to the theoretical foundation of functional differential inclusions with delays. Researchers were encouraged to further explore these outcomes and to *develop computational and numerical methods for approximating the theoretical solutions, thereby extending the applicability of the current research.*

Key words: mild solutions, state-dependent delays, Banach spaces, multivalued maps, fixed points, functional differential inclusions, semi-linear systems

Introduction

Functional differential equations and inclusions play a crucial role in various fields such as biology, physics, and engineering, and have attracted significant attention in recent years.To navigate the extensive literature on functional differential equations, valuable resources include books by Hale (1977), Hale and Lunel (2013), Kolmanovskii & Myshkis, (1999), along with the references contained therein.

Over the past decades, researchers have conducted extensive investigations into the existence and uniqueness of solutions for semi-linear functional differential equations and inclusions. Various types of solutions, including mild, strong, classical, almost periodic, and almost automorphic solutions, have been explored using methodologies such as semigroup theory, fixed point arguments, degree theory, and measures of non-compactness. Noteworthy contributions to this body of work can be found in books by (Ahmed, 1991, Kolmanovskii et al, 1999 & Wu 1996), and the related references such as the articles by (Daoudi et al, 2018; Mohamed & Tayeb, 2019).

In this paper, we consider a system with state-dependent delays of the following form:

$$\begin{cases} x'^{(t)} - A_1 x(t) \in F_1\left(t\,, x\big(t-\mathcal{T}_1(t,x_t)\big), y\big(t-\mathcal{T}_2(t,y_t)\big)\right), t \in J := [0,b] \\ y'(t) - A_2 y(t) \in F_2\left(t, x\big(t-\mathcal{T}_1(t,x_t)\big), y\big(t-\mathcal{T}_2(t,y_t)\big)\right), t \in J := [0,b] \\ x(t) = \varphi_1(t), t \in [-r,0] \\ y(t) = \varphi_2(t), \qquad t \in [-r,0] \\ \qquad x(0) = x(b) \; and \; y(0) = y(b) \end{cases} \tag{1}$$

where the operators $\mathrm{A_i}, \mathrm{i} = 1; 2$ are the infinitesimal generator of a $\mathrm{C_0}$-semigroup $\mathrm{T_i(t)_{t\geq 0}}$ on a Banach space E. b; r are real numbers with b, r>0 .

$\mathrm{F_1}, \mathrm{F_2} : \mathrm{J} \times \mathrm{E} \times \mathrm{E} \to \mathcal{P}(\mathrm{E})$ are multifunctions, and

$\mathcal{T}_\mathrm{i} : [0, \mathrm{b}] \times \mathrm{C}([-\mathrm{r}, 0], \mathrm{E}) \to [0, \mathrm{r}], \mathrm{i} = 1; 2$ are the given continuous functions.

For any function x defined on $[-r;\ b]$ and any $t \in J$, we denote by x_t the element of $C([-r;\ 0], E)$ defined by

$$x_t(\theta) = x(t + \theta), \theta \in [-r, 0].$$

Here, $x_t(\cdot)$ represents the history of the state from the time $t - r$, up to the present time t.

Throughout this paper, the operators A_i, $i = 1, 2$ are the infinitesimal generator of a C_0 −semigroup $T_i(t)_{t\geq 0}$ and there exists $M > 0$ such that

$$\|T(t)\| \leq M \text{ for all } t \in J.$$

$C([-r, b], E)$ is the Banach space of all continuous functions from $[-r;\ b]$ into E with the norm

$$\|x\|_\infty = \sup_{\theta\in[-r,0]} \sup_{t\in[0,b]} \|x(t+\theta)\|.$$

Set $C_r := C([-r, 0], E)$ and $C := C([-r, b], E)$.

This paper is organized as follows: Section 2 briefy recalls some basic definitions and preliminary facts which will be used throughout the following sections.

Section 3 proves the existence of set solutions for a functional differential inclusions system with state-dependent delays with periodic conditions.

Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Definition (2.1) (Daoudi, 2018): A square matrix of real numbers is said to converge to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc $i.e |\lambda| < 1$; for every $\lambda \in \mathbb{C}$ with $\det(M - \lambda I) = 0$, where I denote the unit matrix of $M_{n\times n}(\mathbb{R})$.

Multi-valued analysis

Let (X, d)be a metric space and Y be a subset of X: Denote by

• $\mathcal{P}(X) = \{Y \subset X: Y \neq \emptyset\}$.

• $\mathcal{P}_p(X) = \{Y \in \mathcal{P}(X): Y$ has the property “p”$\}$ where p could be: cl =closed,

b=bounded, cp =compact, cv =convex, etc. Thus,

•$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X): Y\ closed\}$

• $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X): Y\ bounded\}$.

•$\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X): Y \text{ compact}\}$.

• $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X): Y\ convex\}$where X is a Banach space.
• $\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$.

Let $(\mathrm{X}, \mathrm{d}_*)$ be a metric space. Denote by $\mathrm{H}_{\mathrm{d}_*}$the Hausdorff pseudo-metric distance on $\mathrm{P}(\mathrm{X})$ defined as

$$H_{d_*}: \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}, H_{d_*}(A,B) = max\left\{\sup_{a\in A} d_*(a,B), \sup_{b\in B} d_*(A,b),\right\}.$$

where $\mathrm{d}_*(A,b) = \inf_{a\in A} d_*(a,b\ )$ and $d_*(a,B) = \inf_{b\in B} d_*(a,b)$. Then $(\mathcal{P}_{b,cl}(X), H_{d_*})$ is a metric space and $(\mathcal{P}_{\mathrm{cl}}(\mathrm{X}), \mathrm{H}_{\mathrm{d}_*})$ is a generalized metric space. In particular, $\mathrm{H}_{\mathrm{d}_*}$ satisfies the triangle inequality.

Consider the generalized Hausdorff pseudo-metric distance $\mathrm{H}_{\mathrm{d}}: \mathcal{P}(\mathrm{X}) \times \mathcal{P}(\mathrm{X}) \to \mathbb{R}_+^n \cup \{\infty\}$

defined by

$$H_d(A,B) \coloneqq \begin{pmatrix} H_{d_1}(A,B) \\ \cdots \\ H_{d_n}(A,B) \end{pmatrix}.$$

Definition 2.2 (Daoudi, 2018): Let (X, d) be a generalized metric space, and let $N: X \to \mathcal{P}_{cl}(X)$ be a multivalued operator. The operator N is said to be contractive if there exists a matrix $\mathrm{M} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that

$M^k \to 0\ as\ k\ \to \infty$

and

$H_d\big(N(u), N(v)\big) \leq\ Md(u,v), \qquad for\ all\ u, v\ \in\ X.$

Let (X, d) and (Y, ρ) be two metric spaces and $\mathrm{F}: \mathrm{X} \to \mathcal{P}(\mathrm{Y}\ )$ be a multi-valued mapping. Then F is said to be *lower semi-continuous* $(\mathrm{l.s.c.})$ if the inverse image of V by F

$$F^{-1}(V\ ) = \{x \in X: F(x) \cap V\ \neq \emptyset\}$$

is open for any open set V in Y . Equivalently, F is l.s.c. if the core of V by F

$$F^{+1}(V\ ) = \{x \in X: F(x) \subset V\ \}$$

is closed for any closed set V in Y .

Likewise, the map F is called upper semi-continuous$(\mathrm{u.s.c.})$ on X if for each $x_0 \in\ \mathrm{X}$ the set $\mathrm{F}(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $\mathrm{F}(\mathrm{x}_0)$, there exists an open neighborhood M of x_0 such that $\mathrm{F}(\mathrm{M}) \subseteq \mathrm{Y}$. That is, if the set $\mathrm{F}^{-1}(\mathrm{V}\ )$ is closed for any closed set V in Y. Equivalently, F is $\mathrm{u.s.c.}$ if the set $\mathrm{F}^{+1}(\mathrm{V}\ )$is open for any open set V in Y .

The mapping F is said to be completely continuous if it is $\mathrm{u.s.c.}$ and, for every bounded subset $\mathrm{A} \subseteq \mathrm{X}$, $\mathrm{F(A)}$ is relatively compact, i.e., there exists a relatively compact set $\mathrm{K} = \mathrm{K(A)} \subset \mathrm{X}$ such that

$$F(A) = \bigcup \{F(x) : x \in A\} \subset K.$$

Also, F is compact if $\mathrm{F(X)}$ is relatively compact, and it is called locally compact if for each $x \in \mathrm{X}$, there exists an open set U containing x such that $\mathrm{F(U)}$ is relatively compact.

We denote the graph of F to be the set

$Graph(F) = \{(x, y) \in X \times Y, y \in F(x\,)\}$, and we recall the following facts.

Definition 2.3 (Daoudi, 2018): A multivalued map $F: [a, b] \to \mathcal{P}(Y\,)$ is said to be measurable if for every open $U\ \subset Y$; the set

$\mathrm{F}_{+}^{-1}(\mathrm{U}) = \{x \in \mathrm{Y} : \mathrm{F}(x) \subset \mathrm{U}\}$ is Lebesgue measurable.

Definition 2.4 (Djebali et al, 2010): A multi-map F is called a Carathéodory function if

(a) the multi-map $t \to F(t, x)$ is measurable for each $x \in \mathrm{X}$;

(b) for a.e. $\mathrm{t} \in \mathrm{J}$, the map $x \to \mathrm{F}(\mathrm{t}, x)$ is upper semi-continuous.

Furthermore, F is L^1-Carathéodory if it is further locally integrably bounded, i.e., for each positive r, there exists $\mathrm{h_r} \in \mathrm{L}^1(\mathrm{J}, \mathbb{R}^+)$ such that

$\|(\mathrm{t}, x)\|_{\mathcal{P}} \leq \mathrm{h_r}(\mathrm{t}), \qquad \text{for a.e.}\ \mathrm{t} \in \mathrm{J}\ \text{and all}\ |x| \leq \mathrm{r}.$

Lemma 2.5 (Hale, 1977) : The multivalued map $F: [a, b] \to P_{cl}(Y\,)$ is measurable if and only if for each $x \in \mathrm{Y}$, the function $\zeta: [a, b] \to [0, +\infty)$ defined by

$\zeta(t) = dist(x, F(t)) = inf\{\|x - y\| : y \in F(t)\}, t \in [a, b]$, *is* Lebesgue measurable.

The following two lemmas are needed. The first one is the celebrated Kuratowski-Ryll-Nardzewski selection theorem.

Lemma 2.6 (Daoudi, 2018): Let Y be a separable metric space and $F: [a, b] \to \mathcal{P}(Y)$ a measurable multi-valued map with nonempty closed values. Then F has a measurable selection.

Lemma 2.7 (Daoudi, 2018): Let I be a compact interval and E be a Banach space. Let F be an L^1-Carathéodory multi-valued map with $\mathrm{S_{F,y}} \neq \emptyset$, and let Γ be a linear continuous mapping from $\mathrm{L}^1(\mathrm{I}, \mathrm{E})$ to $\mathrm{C}(\mathrm{I}, \mathrm{E})$. Then, the operator

$$\Gamma o S_F : C(I, E) \to \mathcal{P}_{cp,c}(E), \qquad y \to (\Gamma o S_F)(y) = \Gamma(S_{F,y}),$$

is a closed graph operator in $\mathrm{C}(\mathrm{I}, \mathrm{E}) \times \mathrm{C}(\mathrm{I}, \mathrm{E})$, where $\mathrm{S_{F,y}}$ is known as the selectors set from F and given by

$$f \in S_{F,y} = \{\, f \in\ L^1(I, E) : f(t) \in F(t, y(t))\ for\ a.e.t\ \in\ I\}.$$

Lemma 2.8 (Daoudi, 2018): Let I be a compact interval and E be a Banach space. Let F be an L^1-Carathéodory multi-valued map with $\mathrm{S}_{\mathrm{F,y}} \neq \emptyset$, and let Γ be a linear continuous mapping from $\mathrm{L}^1(\mathrm{I},\mathrm{E})$ to $\mathrm{C}(\mathrm{I},\mathrm{E})$. Then, the operator

$$\Gamma o S_F: C(I,E) \to \mathcal{P}_{cp,c}(E), y \to (\Gamma o S_F)(y) = \Gamma\left(S_{F,y}\right),$$

is a closed graph operator in $\mathrm{C}(\mathrm{I},\mathrm{E}) \times \mathrm{C}(\mathrm{I},\mathrm{E})$, where $\mathrm{S}_{\mathrm{F,y}}$ is known as the selectors set from F and given by

$$f \in S_{F,y} = \left\{f \in L^1(I,E): f(t) \in F\big(t,y(t)\big)\, for\ a.e.t \in I\right\}.$$

Lemma 2.9 (Aubin & Frankowska, 1990): If $G: X \to \mathcal{P}_{cp}$is $\mathrm{u.s.c}$, then for any $\lim\limits_{\mathrm{x}\to\mathrm{x}_0} \sup \mathrm{G}(x) = \mathrm{G}(x_0)$.

Lemma 2.10 (Aubin & Frankowska, 1990): Let $(k_n)_{n\in\mathbb{N}} \subset k \subset X$ be a sequence of subsets where K is compact in the separable Banach space X. Then

$$\overline{\mathrm{co}}\left(\lim_{\mathrm{n}\to\infty} \sup \mathrm{k_n}\right) = \bigcap_{\mathrm{N}>0} \overline{\mathrm{co}}\left(\bigcup_{\mathrm{n}\geq\mathrm{N}} \mathrm{k_n}\right),$$

where $\overline{\mathrm{co}}\mathrm{A}$ refers to the closure of the convex hull of A.

Lemma 2.11 (Aubin & Frankowska, 1990): Every semi-compact sequence $L^1([0;\ b], E)$ is weakly compact in $L^1([0;\ b],; E)$.

Lemma 2.12 (Djebali et al 2010): Let E be a normed space and $x_{kk\in\mathbb{N}} \subset E$ be a sequence weakly converging to a limit $x \in E.$ Then there exists a sequence of convex combinations $\mathrm{y_m} = \sum_{\mathrm{k}=1}^{\mathrm{k}=\mathrm{m}} \alpha_{\mathrm{mk}} x_{\mathrm{k}}$ with $\alpha_{\mathrm{mk}} > 0$ for $\mathrm{k} = 1,2,\ldots,\mathrm{m}$ and $\mathrm{y_m} = \sum_{\mathrm{k}=1}^{\mathrm{k}=\mathrm{m}} \alpha_{\mathrm{mk}} = 1$, which converges strongly to x.

Theorem 2.13 (Sinacer et al, 2016): Let (X; d) be a complete generalized metric space and $\mathrm{F}: \mathrm{X} \to \mathcal{P}_{\mathrm{cl,b}}(\mathrm{X})$ a contractive multivalued operator with the Lipschitz matrix M. Then N has a unique fixed point.

Theorem 2.14 (Wu, 1996): Let (X,d) be a complete generalized metric space and $\mathrm{F}: \mathrm{X} \to \mathcal{P}_{\mathrm{cl}}(\mathrm{X})$ be a multivalued map. Assume that there exist $A, B, C \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that

$$H_d\big(F(x), F(y)\big) \leq Ad(x,y) + Bd\big(y, F(x)\big) + Cd\big(x, F(x)\big),$$

where $\mathrm{A}+\mathrm{C}$ converge to zero. Then there exists $\mathrm{x} \in \mathrm{X}$ such that $x \in \mathrm{F}(\mathrm{x})$.

Existence results

In this section, we assume that $1 \in \rho(T(b))$ and give our main existence and uniqueness result for problem (1). Before starting and proving this result, we give the definition of its mild solution.

Definition 3.1 See details in (Djebali et al 2010).

A function $(x, y) \in C \times C$ is said to be a mild solution of problem (1) if $x(t) = \varphi_1(t),\ y(t) = \varphi_2(t), t \in [-r, 0]$ and if there exists $v_1, v_2 \in L^1(J, E)$ such that

$$v_i \in F_i\left(t, x\left(t - \tau_1(t, x_t)\right), y\left(t - \tau_2(t, y_t)\right)\right) \text{ a.e. on J such that}$$

$$x(t) = T_1(t)(I - T_1(b))^{-1} \int_0^b T_1(b - s) v_1(s)\, ds + \int_0^t T_1(t - s)\, v_1 ds,$$

$$y(t) = T_2(t)(I - T_2(b))^{-1} \int_0^b T_2(b - s) v_2(s)\, ds + \int_0^t T_2(t - s)\, v_2 ds,$$

$$x_0 = x_b \text{ and } y_0 = y_b.$$

Assume that the following conditions:

(G_1) $F_i : J \times E \times E \to P_{cp,cv}(E); t \mapsto F_i(t, u, v);\ i = 1,2$ are measurable for each $t \in J$ and $u, v \in C$.

(G_2) There exist functions $l_i, \overline{l_i} \in L^1(J, \mathbb{R}) \to \mathbb{R}_+, i = 1,2$ such that
$H_d\left(F_i(t, u, v) - F_i(t, \tilde{u}, \tilde{v})\right) \le l_i(t)\|u - \tilde{u}\| + \overline{l_i}(t)\|v - \tilde{v}\|$ for every
$t \in J, u, \tilde{u}, v, \tilde{v} \in E$ and
$H_d\left(0, F_i(t, 0,0)\right) \le l_i(t), for\ all\ a.e.\ t \in J\ and\ i = 1,2.$

(G_3) There exists a constant $\rho_i \ge 0, i = 1;\ 2$ such that

$$\|\tau_i(t, u) - \tau_i(t, \tilde{u})\|) \le \rho_i \|u - \tilde{u}\|$$

for all $u;\ \tilde{u} \in E$ and $t \in J$.

(G_4) There exists a constant $k_i \ge 0, i = 1;\ 2$ such that
$H_d(F_i(t;\ u(s_1);\ v(s_1) - F_i(t;\ u(s_2);\ v(s_1)) \le k_1|s_1 - s_2| + k_2|s_1 - s_2|,$
for every $t \in J,\ s_1;\ s_2 \in [-r;\ b]$ and $u;\ v \in E$.

Theorem (3.2): Assume that (G_1), (G_2), (G_3) and (G_4) are satisfied and the matrix

$$\overline{M} = \left(\frac{M^2}{\|I - T(b)\|_\infty} + M\right)\begin{pmatrix} k_1\rho_1 + \|l_1\|_{L^1} & \bar{k}_1\rho_2 + \left\|\bar{l}_1\right\|_{L^1} \\ k_2\rho_1 + \|l_2\|_{L^1} & \bar{k}_2\rho_2 + \left\|\bar{l}_2\right\|_{L^1} \end{pmatrix}$$

converges to zero; then problem (1) has a unique mild solution.

Proof: Consider the operator $N: C \times C \to P(C \times C)$ defined for $(x, y) \in C \times C$ by

$$N(x;\, y) = \left\{ (h_1, h_2) \in C \times C : \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} T_1(t)\big(I - T_1(b)\big)^{-1} \int_0^b T_1(b-s) v_1(s) ds \\ + \int_0^t T_1(t-s) v_1(s) ds \\ T_2(t)\big(I - T_2(b)\big)^{-1} \int_0^b T_2(b-s) v_2(s) ds \\ + \int_0^t T_2(t-s) v_2(s) ds \end{pmatrix}, for\ t \in [0, b] \right\}$$

and $h_1(t) = \varphi_1(t);\ h_2(t) = \varphi_2(t)$ for $t \in [-r;\ 0]$, where

$$v_i \in S_{F_i,x,y} = \Big\{ f \in L^1(J, E) : f(t) \in F_i\Big(t, x\big(t - \tau_1(t, x_t)\big), y\big(t - \tau_2(t, y_t)\big)\Big), a.e.\ t \in J \Big\}.$$

Clearly, the fixed points of the operator N are the solutions of problem (1). Let

$$N_1(x,y)=\left\{h_1\in C: h_1(t)=\left\{\begin{matrix} T_1(t)\big(I-T_1(b)\big)^{-1}\int_0^b T_1(b-s)v_1(s)ds \\ +\int_0^t T_1(t-s)v_1(s)ds\,,\ \ t\in J:=[0,b] \end{matrix}\right\}\right.$$

and

$$\mathrm{N}_2(x,y)=\left\{h_2\in C: h_2(t)=\left\{\begin{matrix} T_2(t)\big(I-T_2(b)\big)^{-1}\int_0^b T_2(b-s)v_2(s)ds \\ +\int_0^t T_2(t-s)v_2(s)ds\,,\ \ t\in J:=[0,b] \end{matrix}\right\}\right..$$

Hence,

$$N(x,y)=\big(N_1(x,y),N_2(x,y)\big)\ \ for\ every\ \ (x,y)\in C\times C.$$

Since for each $(x,\mathrm{y})\in \mathrm{C}\times\mathrm{C}$, the nonlinearity $\mathrm{F_i}$ takes convex values, the selection set $\mathrm{S_{F_i,x,y}}$ is convex, then N has convex values. We show that N satisfies the assumptions of Theorem 2.13.

Let $(\mathrm{x},\mathrm{y}),(\tilde{\mathrm{x}},\tilde{\mathrm{y}})\in \mathrm{C}^2\times\mathrm{C}^2$and $(\mathrm{h_1},\mathrm{h_2})\in \mathrm{N}(\mathrm{x},\mathrm{y})$. Then there exists $\mathrm{v_i}\in \mathrm{F_i}\Big(\mathrm{t},\mathrm{x}\big(\mathrm{t}-\tau_1(\mathrm{t},\mathrm{x_t})\big),\mathrm{y}\big(\mathrm{t}-\tau_2(\mathrm{t},\mathrm{y_t})\big)\Big)$, $\mathrm{i}=1,2$ such that

$$\begin{pmatrix} h_1(t)\\ h_2(t)\end{pmatrix}=\begin{pmatrix} T_1(t)\big(I-T_1(b)\big)^{-1}\int_0^b T_1(b-s)v_1(s)ds \\ +\int_0^t T_1(t-s)v_1(s)ds \\ T_2(t)\big(I-T_2(b)\big)^{-1}\int_0^b T_2(b-s)v_2(s)ds \\ +\int_0^t T_2(t-s)v_2(s)ds \end{pmatrix},for\ t\in[0,b]$$

$(\mathrm{G_2})$ implies that

$$\begin{aligned}
&H_{d_1}\big(F_1(t,x\big(t-\tau_1(t,x_t)\big),y(t-\tau_2(t,y_t)))\\
&\qquad -F_1(t,\tilde{x}\big(t-\tau_1(t,\tilde{x}_t)\big),\tilde{y}(t-\tau_2(t,\tilde{y}_t)))\big)\le\\
&H_{d_1}\big(F_1(t,x\big(t-\tau_1(t,x_t)\big),y(t-\tau_2(t,y_t)))\\
&\qquad -F_1(t,\tilde{x}\big(t-\tau_1(t,\tilde{x}_t)\big),\tilde{y}(t-\tau_2(t,\tilde{y}_t)))\big)+\\
&H_{d_1}\big(F_1(t,\tilde{x}\big(t-\tau_1(t,x_t)\big),\tilde{y}(t-\tau_2(t,y_t)))\\
&\qquad -F_1(t,\tilde{x}\big(t-\tau_1(t,\tilde{x}_t)\big),\tilde{y}(t-\tau_2(t,\tilde{y}_t)))\big)\le\\
&l_1(t)\big(\big\|x\big(t-\tau_1(t,x_t)\big)-\tilde{x}\big(t-\tau_1(t,\tilde{x}_t)\big)\big\|+k_1\|\tau_1(t,x_t)-\tau_1(t,x_t)\|\big)+
\end{aligned}$$

$$\bar{l}_1(t)\left(\left\|y\big(t-\tau_2(t,y_t)\big)-\tilde{y}\big(t-\tau_2(t,y_t)\big)\right\|+\bar{k}_1\|\tau_2(t,y_t)-\tau_2(t,\tilde{y}_t)\|\right)\leq$$
$$l_1(t)\left(\left\|x\big(t-\tau_1(t,xx_t)\big)-\tilde{x}\big(t-\tau_1(t,x_t)\big)\right\|+k_1\rho_1\|x_t-\tilde{x}_t\|\right)+$$
$$\bar{l}_1(t)\left(\left\|y\big(t-\tau_2(t,y_t)\big)-\tilde{y}\big(t-\tau_2(t,y_t)\big)\right\|+\bar{k}_1\rho_2\|y_t-\tilde{y}_t\|\right),\qquad t\in J$$

and

$$H_{d_2}\big(F_1(t,x\,\big(t-\tau_1(t,x_t)\big),y(t-\tau_2(t,y_t)))$$
$$-F_1(t,\tilde{x}\big(t-\tau_1(t,\tilde{x}_t)\big),\tilde{y}(t-\tau_2(t,\tilde{y}_t)))\big)\leq$$
$$H_{d_2}\big(F_1(t,x\big(t-\tau_1(t,x_t)\big),y(t-\tau_2(t,y_t)))$$
$$-F_1(t,\tilde{x}\big(t-\tau_1(t,x_t)\big),\tilde{y}(t-\tau_2(t,y_t)))\big)+$$
$$H_{d_2}\big(F_1(t,\tilde{x}\big(t-\tau_1(t,x_t)\big),\tilde{y}(t-\tau_2(t,y_t)))$$
$$-F_1(t,\tilde{x}\big(t-\tau_1(t,\tilde{x}_t)\big),\tilde{y}(t-\tau_2(t,\tilde{y}_t)))\big)\leq$$
$$l_2(t)\left(\left\|x_t\big(t-\tau_1(t,x_t)\big)-\tilde{x}_t\big(t-\tau_1(t,x_t)\big)\right\|+k_1\|\tau_1(t,x_t)-\tau_1(t,\tilde{x}_t)\|\right)+$$
$$\bar{l}_2(t)\left(\left\|y\big(t-\tau_2(t,y_t)\big)-\tilde{y}\big(t-\tau_2(t,y_t)\big)\right\|+\bar{k}_1\|\tau_2(t,y_t)-\tau_2(t,\tilde{y}_t)\|\right)\leq$$
$$l_2(t)\left(\left\|x\big(t-\tau_1(t,x_t)\big)-\tilde{x}\big(t-\tau_1(t,x_t)\big)\right\|+k_1\rho_1\|x_t-\tilde{x}\,\|\right)+$$
$$\bar{l}_2(t)\left(\left\|y\big(t-\tau_2(t,y_t)\big)-\tilde{y}\big(t-\tau_2(t,y_t)\big)\right\|+\bar{k}_1\rho_2\|y_t-\tilde{y}\|\right),t\in J.$$

Hence, there is some

$$(w,\widetilde{w})\in\ F_1(t,x\big(t-\tau_1(t,x_t)\big),y(t-\tau_2(t,y_t)))\times F_2(t,x\big(t-\tau_1(t,x_t)\big),y(t-\tau_2(t,y_t)))$$

such that for each $t\in J$

$$\|v_1(t)-\overline{w}\|\leq l_1(t)\left(\left\|x\big(t-\tau_1(t,x_t)\big)-\tilde{x}\big(t-\tau_1(t,x_t)\big)\right\|+k_1\rho_1\|x_t-\tilde{x}\|\right)$$
$$+$$
$$\bar{l}_1(t)\left(\left\|y\big(t-\tau_2(t,y_t)\big)-\tilde{y}\big(t-\tau_2(t,y_t)\big)\right\|+\bar{k}_1\rho_2\|y_t-\tilde{y}_t\|\right)$$

and

$$\|v_2(t)-\overline{w}\|\leq l_2(t)\left\|x\big(t-\tau_1(t,x_t)\big)-\tilde{x}\big(t-\tau_1(t,x_t)\big)\right\|+k_1\rho_1\|x_t-\tilde{x}_t\|$$
$$+\bar{l}_2(t)\left\|y\big(t-\tau_2(t,y_t)\big)-\tilde{y}\big(t-\tau_2(t,y_t)\big)\right\|+\bar{k}_1\rho_2\|y_t-\tilde{y}_t\|\big).$$

Consider the multi-valued maps $\cup_i\colon J\to\mathbb{R}, i\ =\ 1;\ 2$ defined by

$$U_1(t)=\{f\in F_1(t,x\big(t-\tau_1(t,x_t)\big),y(t-\tau_2(t,y_t))):$$
$$\|v_1(t)-w\|\leq l_1(t)\left\|x\big(t-\tau_1(t,x_t)\big)-\tilde{x}\big(t-\tau_1(t,x_t)\big)\right\|+k_1\rho_1\|x_t-x_t\|$$
$$+\bar{l}_1(t)\left\|y\big(t-\tau_2(t,y_t)\big)-\tilde{y}\big(t-\tau_2(t,y_t)\big)\right\|+\bar{k}_1\rho_2\|y_t-\tilde{y}_t\|\big)$$

and

$$U_2(t)=\{f\in F_2(t,x\big(t-\tau_1(t,x_t)\big),y(t-\tau_2(t,y_t))):$$
$$\|v_2(t)-w\|\leq l_2(t)\left\|x\big(t-\tau_1(t,x_t)\big)-\tilde{x}\big(t-\tau_1(t,x_t)\big)\right\|+k_1\rho_1\|x_t-\tilde{x}_t\|$$

$$+\bar{l}_2(t)\|y(t-\tau_2(t,y_t))-\tilde{y}(t-\tau_2(t,y_t))\|+\bar{k}_1\rho_2\|y_t-\tilde{y}_t\|).$$

Then $U_i(t)$ is a nonempty set and Theorem III,4.1 in (Daoudi, 2018) tells us that U_i is measurable. Moreover, the multi-valued intersection operator $V_i(\cdot)=U_i(\cdot)\cap F_i\left(\cdot,x(\cdot-\tau_1(\cdot,x)),y(\cdot-\tau_2(\cdot,y))\right)$ is measurable. Therefore, by Lemma 2.6, there exists a function $t\to\bar{v}_i$, which is a measurable selection for V_i, that is

$\tilde{v}_i\in F_i\left(t,x(t-\tau_1(t,x_t)),y(t-\tau_2(t,y_t))\right)$ and

$$\|v_1(t)-\tilde{v}_1(t)\|\le l_1(t)\|x(t-\tau_1(t,x_t))-\tilde{x}(t-\tau_1(t,x_t))\|+k_1\rho_1\|x_t-\tilde{x}_t\|$$
$$+\bar{l}_1(t)\|y(t-\tau_2(t,y_t))-\tilde{y}(t-\tau_2(t,y_t))\|+\bar{k}_1\rho_2\|y_t-\tilde{y}_t\|)$$

and

$$\|v_2(t)-\tilde{v}_2(t)\|\le l_2(t)\|x(t-\tau_1(t,x_t))-\tilde{x}(t-\tau_1(t,x))\|+k_1\rho_1\|x_t-\tilde{x}_t\|$$
$$+\bar{l}_2(t)\|y(t-\tau_2(t,y_t))-\tilde{y}(t-\tau_2(t,y_t))\|+\bar{k}_1\rho_2\|y_t-\tilde{y}_t\|).$$

Define $\tilde{h}_1,\tilde{h}_2$ by

$$\tilde{h}_1(t)=T_1(t)(I-T_1(b))^{-1}\int_0^b T_1(b-s)v_1(s)ds+\int_0^t T_1(t-s)v_1(s)ds,t\in J$$

and

$$\tilde{h}_2(t)=T_2(t)(I-T_2(b))^{-1}\int_0^b T_2(b-s)v_2(s)ds+\int_0^t T_2(t-s)v_2(s)ds\,,t\in J.$$

Then for $t\in J$, there is

$$\|h_1(t)-\tilde{h}_1(t)\|$$
$$\le\frac{M^2}{\|I-T(b)\|}\left(\int_0^b l_1(s)\|x(s-\tau_1(s,x_s))-\tilde{x}(s-\tau_1(s,x_s))\|\right.$$
$$+k_1\rho_1\|x_s-\tilde{x}_s\|+\bar{l}_1(s)\|y(s-\tau_1(s,y_s))-\tilde{y}(s-\tau_2(s,y_s))\|$$
$$\left.+\bar{k}_1\rho_2\|y_s-\tilde{y}_s\|\right)ds$$
$$+M\left(\int_0^t l_1(s)\|x(s-\tau_1(s,x_s))-\tilde{x}(s-\tau_1(s,x_s))\|+k_1\rho_1\|x_s-\right.$$
$$\left.\tilde{x}_s\|+\bar{l}_1(s)\|y(s-\tau_2(s,y_s))-\tilde{y}(s-\tau_2(s,y_s))\|+\bar{k}_1\rho_2\|y_s-\tilde{y}_s\|\right)ds$$

$$\leq\left(\frac{M^2}{\|1-T(b)\|}+M\right)\left(\int_0^b l_1(s)\|x(s-\tau_1(s,x_s))-\tilde{x}(s-\tau_1(s,x_s))\| + k_1\rho_1\|x_s-\tilde{x}_s\| + \bar{l}_1(s)\|y(s-\tau_2(s,y_s))-\tilde{y}(s-\tau_2(s,y_s))\| + \bar{k}_1\rho_2\|y_s-\tilde{y}_s\|)ds\right)$$

$$\leq\left(\frac{M^2}{\|1-T(b)\|}+M\right)\left(k_1\rho_1+\|l_1\|_{L^1},\bar{k}_1\rho_2+\left\|\bar{l}_1\right\|_{L^1}\right)(\|x-\tilde{x}\|_\infty,\|y-\tilde{y}\|_\infty).$$

Thus,

$$\left\|h_1-\tilde{h}_1\right\|_\infty\leq\left(\frac{M^2}{\|1-T(b)\|}+M\right)\left(k_1\rho_1+\|l_1\|_{L^1},\bar{k}_1\rho_2 + \left\|\bar{l}_1\right\|_{L^1}\right)(\|x-\tilde{x}\|_\infty,\|y-\tilde{y}\|_\infty).$$

By an analogous relation, one finally arrives at the estimate

$$H_{d_1}(N_1(x,y),N_1(\tilde{x},\tilde{y})) \leq\left(\frac{M^2}{\|1-T(b)\|}+M\right)\left(k_1\rho_1+\|l_1\|_{L^1},\bar{k}_1\rho_2+\left\|\bar{l}_1\right\|_{L^1}\right)(\|x-x\|_\infty,\|y-\tilde{y}\|_\infty).$$

Similarly, there is

$$H_{d_2}(N_2(x,y),N_2(\tilde{x},\tilde{y})) \leq\left(\frac{M^2}{\|1-T(b)\|}+M\right)\left(k_2\rho_1+\|l_2\|_{L^1},\bar{k}_2\rho_2 + \left\|\bar{l}_2\right\|_{L^2}\right)(\|x-\tilde{x}\|_\infty,\|y-\tilde{y}\|_\infty)$$

Therefore,

$$H_d(N(x,y),N(\tilde{x},\tilde{y}))\leq\overline{M}\begin{pmatrix}\|x-\tilde{x}\|_\infty\\\|y-\tilde{y}\|_\infty\end{pmatrix}, for\ all (x,y),(\tilde{x},\tilde{y})\in C^2\times C^2.$$

Thus, by Theorem 2.13, the operator N has a unique fixed point which is a unique mild solution to problem (1).

Conclusion

This paper, using the recent Perov fixed point theorem technique on a Banach space, presents an existence result for a mild solution to a semi-linear operator system of periodic functional differential inclusions.

Researchers are encouraged to explore these findings further and develop computational and numerical methods to approximate the results, providing an alternative approach to advancing the current study.

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Sistema de inclusión diferencial funcional semilineal periódico con retardos dependientes del estado

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CAMPO: Matemáticas
TIPO DE ARTÍCULO: artículo científico original

Resumen:

Introducción/objetivo: El estudio abordó la existencia y singularidad de soluciones suaves para una clase de sistemas de inclusión diferenciales funcionales semilineales periódicos con retardos dependientes del estado. Estos sistemas son de creciente interés debido a su amplia gama de aplicaciones en teoría de control, sistemas biológicos y modelos de ingeniería que involucran retrasos de tiempo e incertidumbres.
Métodos: La investigación se realizó en el marco de los espacios de Banach. Se aplicó el teorema del punto fijo de Perov, conocido por su eficacia en el manejo de sistemas con múltiples componentes, para establecer la existencia de soluciones suaves. El enfoque analítico incorporó técnicas adecuadas para inclusiones diferenciales y teoría de operadores, con especial atención a sistemas influenciados por retardos dependientes del estado.
Resultados: Se obtuvieron varios resultados relativos a la existencia y unicidad de soluciones suaves para el sistema considerado. La aplicación del teorema del punto fijo de Perov permitió derivar las condiciones suficientes para la existencia de dichas soluciones. Además, se tuvo en cuenta la naturaleza periódica del sistema, asegurando que las soluciones se ajustaran a la estructura temporal prescrita.
Conclusión: Los hallazgos confirmaron que, bajo supuestos específicos, el sistema admite al menos una solución moderada. Los resultados contribuyeron a la fundamentación teórica de las inclusiones diferenciales funcionales con retardos. Se alentó a los investigadores a explorar más a fondo estos resultados y a desarrollar métodos computacionales y numéricos para aproximar las soluciones teóricas, ampliando así la aplicabilidad de la investigación actual.

Palabras claves: soluciones suaves, retardos dependientes del estado, espacios de Banach, aplicaciones multivaluadas, puntos fijos, inclusiones diferenciales funcionales, sistemas semilineales

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Периодическая полулинейная функционально-дифференциальная система включения с зависящими от состояния задержками

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РУБРИКА ГРНТИ: 27.01.00 Общие вопросы математики
ВИД СТАТЬИ: оригинальная научная статья

Резюме:

Введение/цель: В данном исследовании рассматривались существование и уникальность мягких решений для класса периодических полулинейных систем функционально-дифференциального включения с задержками, зависящими от состояния. Такие системы вызывают большой интерес из-за их широкого спектра применения в теории управления, биологических системах и инженерных моделях, связанных с временными задержками и неопределенностями.

Методы: *Исследование проводилось в рамках банахового пространства. Теорема Перова о неподвижной точке, отличающаяся эффективностью в работе с многокомпонентными системами, была применена для выявления слабых решений. Аналитический подход включал методы, подходящие для дифференциальных включений и теории операторов, с особым упором на системы, подверженным задержкам, зависящим от состояния.*

Результаты: В ходе исследования было получено несколько результатов, касающихся существования и уникальности слабых решений по анализируемой системе. Применение теоремы Перова о неподвижной точке помогло выявить условия, при которых такие решения могут использоваться. Помимо того, был учтен периодический характер системы, что гарантировало соответствие решений предписанной временной структуре.

Выводы: Результаты подтвердили, что при определенных обстоятельствах система допускает хотя бы одно мягкое решение. Результаты способствовали теоретическому обоснованию функционально-дифференциальных включений с задержками. Исследователям было предложено продолжить изучение полученных результатов и разработать численные методы для аппроксимации теоретических решений, тем самым расширив применимость данного исследования.

Ключевые слова: мягкие решения, зависящие от состояния задержки, банахово пространство, многозначные отображения, неподвижные точки, функционально-дифференциальные включения, полулинейные системы.

Периодични семилинеарни функционални систем инклузије са кашњењима зависним од стања

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Сажетак:

Увод/Циљ: У овој студији обрађује се постојање и јединственост слабих решења за класу периодичних семилинеарних функционалних диференцијалних система инклузије са кашњењима зависним од стања. Све је веће интересовање за ове системе због њиховог широког спектра примене у теорији управљања, биолошким системима, као и инжењерским моделима који укључују временска кашњења и неизвесности.
Методе: Истраживање је спроведено унутар Банахових простора. Теорема о фиксној тачки Перова, позната по својој ефикасности при раду са вишекомпонентним системима, примењена је како би се утврдило постојање слабих решења. Аналитички приступ укључивао је технике погодне за диференцијалне инклузије и теорију оператора, с посебним фокусом на системе под утицајем кашњења зависних од стања.
Резултати: Добијено је неколико резултата који се односе на постојање и јединственост слабих решења за разматрани систем. Примена теореме о фиксној тачки Перова омогућила је извођење довољних услова под којима таква решења постоје. Штавише, узета је у обзир периодична природа система, што је обезбедило да се решења придржавају прописане временске структуре.
Закључак: Налази су потврдили да, под специфичним претпоставкама, систем дозвољава најмање једно слабо решење. Резултати су допринели теоријској основи функционалних диференцијалних инклузија са кашњењима. Истраживачи се подстичу да даље испитују ова решења и да развијају рачунарске и нумеричке методе за апроксимацију теоријских решења, што би омогућило ширу примену овог истраживања.

Кључне речи: слаба решења, кашњења зависна од стања, Банахови простори, вишевредносне мапе, непомичне тачке, функционалне диференцијалне инклузије, семилинеарни системи

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