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RESTRICTED AND EXTENDED THETA OPERATIONS OF SOFT SETS: NEW RESTRICTED AND EXTENDED SOFT SET OPERATIONS

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ABSTRACT

Since its introduction by Molodtsov in 1999, soft set theory has gained widespread recognition as a method for addressing uncertainty-related issues and modeling uncertainty. It has been used to solve several theoretical and practical issues. Since its introduction, the central idea of the theory-soft set operations-has captured the attention of scholars. Numerous limited and expanded businesses have been identified, and their attributes have been scrutinized thus far. We present a detailed analysis of the fundamental algebraic properties of our proposed restricted theta and extended theta operations, which are unique restricted and extended soft set operations. We also investigate these operations' distributions over various kinds of soft set operations. We demonstrate that, when coupled with other types of soft set operations, the extended theta operation forms numerous significant algebraic structures, such as semirings in the collection of soft sets over the universe, by taking into account the algebraic properties of the extended theta operation and its distribution rules. This theoretical subject is very important from both a theoretical and practical perspective since soft sets' operations form the foundation for numerous applications, including cryptology and decision-making procedures.

Keywords: Soft sets, Soft set operations, Restricted theta operation, Extended theta operation.

INTRODUCTION

The real world is filled with a lot of uncertainty. Conventional mathematical reasoning is unable to tackle these issues. More scientific investigation that goes beyond the capability of currently accessible methodologies has been necessary to dispel these uncertainties. In this sense, Pascal and Fermat created the theory of probability in the early 17th century when they conducted an analytical study of the uncertainty problem. In the early 1800s, a large number of scientists investigated uncertainty.

Many values were discovered as a result of Heisenberg's 1920 explanation, which was the first to explain uncertainty. Early in the 1930s, Lukaisewicz developed the first three-valued logic system. A few theories that may be used to describe uncertainty include probability theory, interval mathematics, and fuzzy set theory; however, each of these theories has drawbacks of its own. Thus, the concept of "Soft Set" was first proposed by Molodtsov (1999) and has nothing to do with how the membership function evolved. While soft set theory utilizes a set-valued function instead of a real-valued one, fuzzy set theory aims to eliminate ambiguity. This idea has been successfully applied in several mathematical fields since its conception, such as Riemann integration, Perron integration analysis, game theory, probability theory, and measurement theory.

Soft set operations were first studied by Maji et al. (2003) and Pei and Miao (2005). Ali et al. introduced a number of soft set operations (2009), including restricted and extended soft set operations. In their work on soft sets, Sezgin & Atagün (2011) established and gave the characteristics of the restricted symmetric difference of soft sets. They also explored the principles of soft set operations and gave illustrations of how they relate to one another. A thorough examination of the algebraic structures of soft sets was carried out by Ali et al. (2011). A number of academics were interested in soft set operations and conducted extensive studies on the subject in (Yang, 2008; Neog & Sut, 2011; Fu, 2011; Ge & Yang, 2011; Singh & Onyeozili, 2012a; Singh & Onyeozili, 2012b; Singh & Onyeozili, 2012c; Singh & Onyeozili, 2012d; Husain et al., 2018).

In recent years, a wide variety of novel soft set operations have been implemented. The idea and characteristics of the soft binary piecewise difference operation in soft sets were initially presented and examined by Eren & Çalışıcı (2019). Sezgin et al. (2019) introduced the extended difference of soft sets, while Stojanovic (2021) characterized the extended symmetric difference along with its properties. Furthermore, a comprehensive examination of restricted and extended symmetric difference operations was carried out by Sezgin & Çağman (2024). Sezgin et al. (2023c) worked on numerous new binary set operations and defined several more, inspired by the work of Çağman (2021), who introduced two new complement

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operations to the literature. Using this method, Aybek (2024) proposed several new restricted and extended soft set operations. Three authors, Akbulut (2024), Demirci (2024), and Sarıalioğlu (2024), focused on complementary extended soft sets operations in their attempts to alter the structure of extended operations in soft sets. Other types of soft set operations, complementary soft binary piecewise operations, were further investigated by (Sezgin & Aybek, 2023; Sezgin & Akbulut, 2023; Sezgin & Dagtoros, 2023; Sezgin & Demirci, 2023; Sezgin & Sarıalioğlu, 2024; Sezgin & Yavuz, 2023a; Sezgin et al., 2023a; Sezgin & Atagün, 2023; Sezgin & Çağman, 2024). In addition, Sezgin & Çalışıcı (2024) carried out a comprehensive analysis of the soft binary piecewise difference operation, while Sezgin & Yavuz (2023) and Yavuz (2024) investigated other soft binary piecewise operations.

Classifying algebraic structures and finding, representing, and drawing inferences from their common features are the goals of abstract algebra. The name of the abstract algebra used in this area of mathematics is due to this. Mathematicians have studied algebraic structures for millennia because they offer an abstract and universal foundation for learning and understanding mathematical topics. Many branches of mathematics depend on algebraic structures. There are several significant applications of algebraic structures, such as rings, groups, and fields, in mathematics as well as other disciplines like computer science and physics. The foundation for comprehending increasingly difficult mathematical ideas is laid by the frameworks of algebraic geometry (the study of multivariable polynomial solutions), algebraic topology, modular arithmetic, physics, number theory, and computer graphics, among other extremely significant subjects. Moreover, a foundation for comprehending and researching a broad variety of mathematical objects and their relationships is provided by mathematical structures.

Groups have applications in physics, chemistry, and cryptography and are used to study symmetries, rotations, and transformations in mathematical contexts. Studying the symmetries of fascinating geometric objects and forms requires an understanding of fundamental groups and their representations as group transformations, which are fundamental algebraic structures. Abstract algebra, coding theory, and number theory all make use of rings. Geometry and other mathematical topics require a solid understanding of field algebra. Engineering, quantum physics, and linear algebra all employ vector spaces. Algebra is used in computer science, physics, and mathematical reasoning. Both representation theory and abstract algebra make use of modules. Moreover, abstract algebra, which examines the shared structures and common features of many algebraic systems, is centered on the study of algebraic structures. By knowing these systems' features, mathematicians may create new theories, solve challenging problems, and apply ideas to a variety of mathematical, scientific, and technical fields. Additionally, special cases of algebraic structures are frequently provided in applications, which make it easier to look at more general cases and help make sense of specific ones.

Near-rings, semirings, and semifields are a few of the most well-known binary algebraic structures, which are the generalizations of rings. For a very long time, academics have been keen to understand more about this subject. Ever since Vandiver (1935) introduced the concept of semirings, a number of researchers have studied it. Semirings are very important in mathematics and have many applications, according to Vandiver (1935). In addition to its significance in geometry, semirings have several applications in the information sciences and practical mathematics (Vandiver, 1935). Semirings are important in pure mathematics and geometry, and they are useful in many other fields as well (Ghosh, 1996; Wechsler, 1978; Golan, 1999; Hebisch & Weinert, 1998; Mordeson & Malik, 2002; Kolokoltsov & Maslov, 1997; Hopcroft & Ullman, 1979; Beasley & Pullman, 1988; Beasley & Pullman, 1992).

The categorization of algebraic structures according to the properties of the operation is one of the most important problems in algebraic mathematics. We may suggest new operations on soft sets, examine their properties, and take into account the algebraic structures they form in the collection of soft sets in order to further our grasp of this subject. Thus far, four extended soft set operations (extended intersection, union, difference, and symmetric difference for soft sets) and four limited soft set operations (restricted intersection, union, difference, and symmetric difference) have been presented.

Our goal is to make a significant contribution to the field of soft set theory by proposing a new restricted and extended soft set operation for soft set theory, which we call "restricted theta operation and extended theta operation of soft sets" and closely examining the algebraic structures associated with them and other soft set operations in the collection of soft sets. With the introduction of the so-called new operations in soft sets, an understanding of the underlying algebraic structures is crucial.

This study is organized as follows: Section 2 serves as a reminder of the basic ideas behind soft sets and other algebraic structures. In Section 3, the new soft set operations are defined. A detailed analysis is conducted on the algebraic characteristics of the theta operation and extended theta operation. Furthermore, we study how these novel soft set operations distribute over the existing soft set operations. Considering the distribution laws and the algebraic characteristics of the soft set operations, an extensive analysis of the algebraic structures formed in the set of soft sets over the universe using these operations is presented. Our demonstration reveals that the collection of soft sets throughout the universe forms several significant algebraic structures, including semirings. A comprehensive analysis expands on our knowledge of the applications and consequences of soft set theory across several fields. In the conclusion section, we discuss the significance of the study's findings and their potential applications.

PRELIMINARIES

This section covers several algebraic structures as well as some basic ideas in soft set theory.

Definition 1. (Molodtsov, 1999) Let U be the universal set, E be the parameter set, P(U) be the power set of U, and T⊆ E. A pair (F, T) is called a soft set on U. Here, F is a function given by $F: T \to P(U)$.

Throughout this paper, the collection of all the soft sets over U (no matter what the parameter set is) is designated by $S_E(U)$ and $S_T(U)$ denotes the collection of all soft sets over U with a fixed parameter set T, where T is a subset of E.

Definition 2. (Ali et al., 2011) Let (F,T) be a soft set over U. If (x)=∅ for every x∈T, then the soft set (F,T) is called a null soft set with respect to K, denoted by $\emptyset_K$. Similarly, let (F,E) be a soft set over U. If F(x)=∅ for every x∈E, then the soft set (F,E) is called a null soft set with respect to E, denoted by $\emptyset_E$ (Ali et al., 2009). A soft set with an empty parameter set is denoted as $\emptyset_\emptyset$. It is obvious that $\emptyset_\emptyset$ is the only soft set with an empty parameter set.

Definition 3. (Ali et al., 2009) Let (F,T) be a soft set over U. If F(x)=U for every x∈T, then the soft set (F,T) is called a relative whole soft set with respect to T, denoted by U_T. Similarly, let (F,E) be a soft set over U. If F(x)=U for every x∈E, then the soft set (F,E) is called an absolute soft set, and denoted by U_E.

Definition 4. (Pei & Miao; 2005) Let (F,T) and (G,Y) be soft sets over U. If T⊆Y and F(x) ⊆G(x) for every x∈T, then (F,T) is said to be a soft subset of (G,Y), denoted by (F,T)$\tilde{\subseteq}$(G,Y). If (G,Y) is a soft subset of (F,T), then (F,T) is said to be a soft superset of (G,Y), denoted by (F,T)$\tilde{\supseteq}$(G,Y). If (F,T)$\tilde{\subseteq}$(G,Y) and (G,Y)$\tilde{\subseteq}$(F,T), then (F,T) and (G,Y) are called soft equal sets.

Definition 5. (Ali et al., 2009) Let (F,T) be a soft set over U. The relative complement of (F,T), denoted by $(F,T)^r=(F^r,T)$, is defined as follows: $F^r(x)=U-F(x)$, for every x∈T.

Çağman (2021), introduced two new complements as the inclusive complement and the exclusive complement, which we denote as + and θ, respectively. For two sets X and Y, these binary operations are defined as X+Y=X'∪Y and XθY=X'∩Y'. Sezgin et al. (2023c) investigated the relationship between these two operations and also introduced three new binary operations: For two sets X and Y, these new operations are defined as X*Y=X'∪Y', XγY= X'∩Y, X$\boldsymbol{\lambda}$Y=X∪Y' (Sezgin et al., 2023c). Let "⋈" be used to represent the set operations (i.e., here, ⋈ can be ∩, ∪,\, Δ, +,θ, *, λ,γ). Then, all types of soft set operations are defined as follows:

Definition 6. (Ali et al., 2009; Sezgin & Atagün, 2011; Ali et al., 2011; Aybek, 2024) Let (F,T) and (G,Y) be two soft sets over U. The restricted ⋈ operation of (F,T) and (G,Y) is the soft set (H, Z), denoted by (F,T) $\bowtie_R$ (G,Y)= (H,Z), where Z = T ∩ Y≠ ∅ and for every x∈ Z, H(x) = F(x) ⋈G(x). Here, if Z= T ∩ Y = ∅, then (F, T) $\bowtie_R$(G, Y)= $\emptyset_\emptyset$.

Definition 7. (Maji et al., 2003; Ali et al., 2009; Sezgin et al., 2019; Stojanavic, 2021; Aybek, 2024) Let (F, T) and (G, Y) be two soft sets over U. The extended ⋈ operation (F, T) and (G,Y) is the soft set (H,Z), denoted by (F, T) $\bowtie_\varepsilon$(G, Y) = (H, Z), where Z= T ∪ Y, and for every x∈ Z,

$$H(x)=\begin{cases} F(x), & x\in T-Y\\ G(x), & x\in Y-T\\ F(x)\bowtie G(x), & x\in T\cap Y\end{cases}$$

Definition 8. (Demirci, 2024; Sarıalioğlu, 2024; Akbulut, 2024) Let (F,T) and (G, Y) be two soft sets over U. The complementary extended $\bowtie_\varepsilon$ operation (F,T) and (G,Y) is the soft set (H,Z), denoted by (F, T) $\overset{*}{\bowtie}_\varepsilon$(G, Y) = (H, Z), where Z = T ∪ Y, and for every x∈ Z,

$$H(x)=\begin{cases} F'(x), & x\in T-Y\\ G'(x), & x\in Y-T\\ F(x)\bowtie G(x), & x\in T\cap Y\end{cases}$$

Definition 9. (Çalışıcı & Eren, 2019; Sezgin & Yavuz, 2023b; Sezgin & Çalışıcı, 2024, Yavuz, 2024) Let (F,T) and (G,Y) be two soft sets on U. The soft binary piecewise ⋈ operation of (F,T) and (G,Y) is the soft set (H,T), denoted by $(F,T)\underset{\bowtie}{\sim}(G,Y)=(H,T)$, where for every x∈ T,

$$H(x)=\begin{cases} F(x), & x\in T-Y\\ F(x)\bowtie G(x), & x\in T\cap Y\end{cases}$$

Definition 10. (Sezgin & Demirci, 2023; Sezgin & Aybek, 2023; Sezgin et al. 2023a, 2023b; Sezgin & Atagün, 2023; Sezgin & Yavuz, 2023a; Sezgin & Dagtoros, 2023; Sezgin & Çağman, 2024; Sezgin & Sarıalioğlu, 2024; Sezgin & Sarıalioğlu, 2024) Let (F,T) and (G,Y) be two soft sets on U. The complementary soft binary piecewise ⋈ operation of (F,T) and (G,Y) is the soft set (H,T), denoted by $(F,T)\overset{*}{\underset{\bowtie}{\sim}}(G,Y)=(H,T)$, where for every x∈ T,

$$H(x)=\begin{cases} F'(x), & x\in T-Y\\ F(x)\bowtie G(x), & x\in T\cap Y\end{cases}$$

For more about soft sets, we refer to (Mahmood et al. 2018; Jana et al., 2019; Muştuoğlu et al., 2016; Sezer et al., 2015b; Sezer, 2014; Sezgin, 2016; Atagün & Sezgin, 2018; Sezgin, 2018; Sezgin et al, 2017; Sezgin et al., 2022; Lawrence & Manoharan, 2023; Jabir et al. 2024).

Definition 11. (Clifford, 1954) Let (S, ⋆) be an algebraic structure. An element s ∈S is called idempotent if $s^2=s$. If $s^2=s$ for every s∈S, then the algebraic structure (S,⋆) is said to be idempotent. An idempotent semigroup is called a band, an idempotent and commutative semigroup is called a semilattice, and an idempotent and commutative monoid is called a bounded semilattice.

In a monoid, although the identity element is unique, a semigroup/groupoid can have one or more left identities; however, if it has more than one left identity, it does not have a right identity element, thus it does not have an identity element. Similarly, a semigroup/groupoid can have one or more right identities; however, if it has more than one right identity, it does not have a left identity element, thus it does not have an identity element (Kilp et al., 2001).

Similarly, in a group, although each element has a unique inverse, in a monoid, an element can have one or more left inverses; however, if an element has more than one left inverse, it does not have a right inverse, thus it does not have an inverse. Similarly, in a monoid, an element can have one or more right inverses; however, if an element has more than one right inverse, it does not have a left inverse, thus it does not have an inverse (Kilp et al., 2001).

Definition 12. Let S be a non-empty set, and let "+" and "⋆" be two binary operations defined on S. If the algebraic structure (S, +, ⋆) satisfies the following properties, then it is called a semiring:

i. (S, +) is a semigroup.
ii. (S, ⋆) is a semigroup,
iii. For every x, y, z ∈S, x⋆(y + z) = x⋆y + x⋆z and (x +y) ⋆z = x⋆z + y⋆z.

If for every x,y∈S, x+y=y+z, then S is called an additive commutative semiring. If for every x,y∈S, x⋆y=y⋆x, then S is called a multiplicative commutative semiring. If there exists an element 1∈S such that x⋆1=1⋆x=x for every x∈S (multiplicative identity), then S is called semiring with unity. If there exists 0∈S such that for every x∈S, 0⋆x=x⋆0=0 and 0+x=x+0=x, then 0 is called the zero of S. A semiring with commutative addition and a zero element is called a hemiring (Vandiver, 1934). We refer to Pant et al. (2024) for the possible implications of network analysis and graph applications with regard to soft sets, which are defined by the divisibility of determinants.

RESTRICTED AND EXTENDED THETA OPERATION

The new restricted theta and extended theta operations for soft sets are presented in this section. By examining the distributive laws across various types of soft sets, it also talks about their algebraic features and connections with other soft set activities. Examining these operations' algebraic structures in the $S_E(U)$ set in conjunction with other specific kinds of soft set operations yields some significant findings.

Restricted Theta Operation and Its Properties

Definition 13. Let (F,T) and (G,Z) be soft sets over U. The restricted theta of (F,T) and (G,Z), denoted by $(F,T)\theta_R(G,Z)$, is defined as $(F,T)\theta_R(G,Z)=(H,C)$, where C=T∩Z, and if C=T∩Z≠∅, then for every α∈C,

$$H(\alpha)=F(\alpha)\theta G(\alpha)=F'(\alpha)\cap G'(\alpha);$$

if C= T∩Z=∅, then (F,T) θ_R (G,Z)=(H, C)= $\emptyset_\emptyset$.

Since the only soft set with empty parameter set is $\emptyset_\emptyset$, if C=T∩Z= ∅, then it is obvious that (F,T) θ_R (G,Z)= $\emptyset_\emptyset$. Thus, in order to define the restricted theta operation of (F,T) and (G,Z), there is no condition that T∩Z≠ ∅.

Example 1. Let E={e_1,e_2,e_3,e_4}be the parameter set, T={e_1,e_3} and Z={e_2,e_3,e_4} be subsets of E, U={h_1,h_2, h_3,h_4, h_5} be the universal set, (F,T) and (G,Z) be the soft sets over U as (F,T)={(e_1,{h_2,h_5), (e_3,{h_1,h_2,h_5})}, (G,Z)={(e_2,{h_1,h_4,h_5}),{(e_3,{h_2,h_3,h_4}),(e_4, h_3,h_5})}.

Here let (F,T)θ_R(G,Z)=(H,T∩ Z), where for every α ∈ T ∩ Z={e_3}.Thus,H(e_3)=F'(e_3) ∩G'(e_3)={h_3,h_4}∩{h_1, h_5}={ h_1, h_3, h_4, h_5}. Hence, (F,T)θ_R(G,Z)={(e_3, ∅)}

Theorem 1. Let (F,T), (G,Z), (H,M), (G,T), (H,T), (K,V) and (L,V) be soft sets over U. Then, we have the followings:

1) The set $S_E(U)$ is closed under θ_R.
2) [(F,T) θ_R(G,Z)] θ_R(H,M) ≠ (F,T) θ_R[(G,Z) θ_R(H,M)].
3) [(F,T) θ_R(G,T)] θ_R(H,T)≠(F,T) θ_R[(G,T) θ_R(H,T)].
4) (F,T) θ_R (G,Z)=(G,Z) θ_R(F,T).
5) (F,T) θ_R(F,T)= $(F,T)^r$.
6) (F,T) $\theta_R\emptyset_T$=$\emptyset_T\theta_R$(F,T)=$(F,T)^r$.
7) (F,T) θ_R $\emptyset_M$=$\emptyset_M\theta_R$ (F,T)=$(F,T\cap M)^r$.
8) (F,T)θ_R $\emptyset_E$=$\emptyset_E$ θ_R (F,T)= $(F,T)^r$.
9) (F,T)$\theta_R\emptyset_\emptyset$=$\emptyset_\emptyset\theta_R$(F,T)= $\emptyset_\emptyset$.
10) (F,T)$\theta_R U_T$= $U_T\theta_R$(F,T)= $\emptyset_T$.
11) (F,T)$\theta_R U_M$=U_M θ_R(F,T)= $\emptyset_{T\cap M}$.
12) (F,T)$\theta_R U_E$=$U_E\theta_R$(F,T)= $\emptyset_T$.
13) (F,T) θ_R $(F,T)^r$=$(F,T)^r$ θ_R(F,T)= $\emptyset_T$.
14) $[(F,T)\ \theta_R(G,Z)]^r$=(F,T) $\cup_R$ (G,Z).
15) (F,T)θ_R(G,T)= U_T if and only if $(F,T)=\emptyset_T$ and $(G,T)=\emptyset_T$.
16) $\emptyset_{T\cap Z}$ ⫅(F,T) θ_R(G,Z) and (F,T) θ_R(G,Z) ⫅ U_T and (F,T) θ_R(G,Z) ⫅ U_Z.
17) (F,T)θ_R(G,Z) ⫅ $(F,T)^r$ and (F,T)θ_R(G,Z) ⫅$(G,Z)^r$.
18) If (F,T) ⫅ (G, Z), (F,T) θ_R(G,Z) =$(G,T)^r$.
19) If (F,T) ⫅ (G, T), then (G,T) θ_R(H,Z) ⫅(F,T) θ_R(H,Z) and (H,Z) θ_R (G,T) ⫅(H,Z) θ_R(G,T).
20) If (G,T) θ_R(H,Z) ⫅(F,T)θ_R(H,Z), then (F,T) ⫅ (G, T) needs not be true. That is, the converse of Theorem 1 (19) is not ture.
21) If (F,T) ⫅ (G, T) and (K,V) ⫅ (L, V), (G,T) θ_R(L,V) ⫅(F,T) θ_R (K,V). Similarly, (L,V) θ_R(G,T) ⫅(K,V) θ_R (F,T).
22) (F,T) θ_R (G,Z) ⫅(F,T) $*_R$ (G,Z) and (G,Z) θ_R (F,T) ⫅(G,Z) $*_R$ (F,T).

Proof. 1) It is clear that θ_R is a binary operation in $S_E(U)$. That is,

$$\theta_R: S_E(U) \times S_E(U) \to S_E(U)$$

$$((F,T), (G,Z)) \rightarrow (F,T)\theta_R(G,Z) = (H, T\cap Z)$$

Similarly,

$$\theta_R: S_T(U) \times S_T(U) \rightarrow S_T(U)$$
$$((F,T), (G,T)) \rightarrow (F,T)\theta_R(G,T) = (H, T\cap T) = (H,T)$$

That is, let T be a fixed subset of the set E and (F,T) and (G,T) be elements of $S_T(U)$, then so is $(F,T)\theta_R(G,T)$. Namely, $S_T(U)$ is closed under θ_R either.

2) Let $(F,T)\theta_R(G,Z)=(S,T\cap Z)$, where for every $\alpha\in T\cap Z$, $T(\alpha)=F'(\alpha)\cap G'(\alpha)$. Let $(S,T\cap Z)\theta_R(H,M)=(R,(T\cap Z)\cap M))$, where for every $\alpha\in(T\cap Z)\cap M$, $R(\alpha)=T'(\alpha)\cap H'(\alpha)$. Thus,

$$R(\alpha)=[F(\alpha)\cup G(\alpha)]\cap H'(\alpha)$$

Let $(G,Z)\theta_R(H,M)=(K,Z\cap M)$, where for every $\alpha\in Z\cap M$, $K(\alpha)=G'(\alpha)\cap H'(\alpha)$. Let $(F,T)\theta_R(K,Z\cap M)=(S,T\cap(Z\cap M))$, where for every $\alpha\in T\cap(Z\cap M)$, $S(\alpha)=F'(\alpha)\cap K'(\alpha)$. Thus,

$$S(\alpha)=F'(\alpha)\cap[G(\alpha)\cup H(\alpha)]$$

Thus, $(R,(T\cap Z)\cap M)\neq(S,T\cap(Z\cap M))$. That is, in $S_E(U)$, the operation θ_R is not associative. Here, it is obvious that if $T\cap Z=\emptyset$ or $Z\cap M=\emptyset$ or $T\cap M=\emptyset$, then since both sides of the equality is $\emptyset_\emptyset$, the operation θ_R is associative under these conditions.

3) Let $(F,T)\theta_R(G,T)=(K,T)$, where for every $\alpha\in T\cap T=T$, $K(\alpha)=F'(\alpha)\cap G'(\alpha)$. Let $(K,T)\theta_R(H,T)=(R,T)$, where for every $\alpha\in T\cap T=T$, $R(\alpha)=K'(\alpha)\cap H'(\alpha)$. Hence,

$$R(\alpha)=[F(\alpha)\cup G(\alpha)]\cap H'(\alpha)$$

Let $(G,T)\theta_R(H,T)=(L,T)$, where for every $\alpha\in T\cap T$, $L(\alpha)=G'(\alpha)\cap H'(\alpha)$. Let $(F,T)\theta_R(L,T)=(N,T)$, where for every $\alpha\in T\cap T$, $N(\alpha)=F'(\alpha)\cap L'(\alpha)$. Hence,

$$N(\alpha)=F'(\alpha)\cap[G(\alpha)\cup H(\alpha)]$$

Thus, $(R,T)\neq(N,T)$. That is, θ_R is not associative in the collection of soft sets with a fixed parameter set.

4) Let $(F,T)\theta_R(G,Z)=(H,T\cap Z)$, where for every $\alpha\in T\cap Z$, $H(\alpha)=F'(\alpha)\cap G'(\alpha)$. Let $(G,Z)\theta_R(F,T)=(S,Z\cap T)$, where for every $\alpha\in Z\cap T$, $S(\alpha)=G'(\alpha)\cap F'(\alpha)$. Thus,

$$(F,T)\,\theta_R\,(G,Z)=(G,Z)\,\theta_R(F,T).$$

That is, θ_R is commutative in $S_E(U)$. Here it is obvious that if $T\cap Z=\emptyset$, then since both sides is $\emptyset_\emptyset$, θ_R is commutative in $S_E(U)$ under this condition. Moreover, it is evident that $(F,T)\theta_R(G,T)=(G,T)\theta_R(F,T)$, namely, θ_R is commutative in the collection of soft sets with a fixed parameter set.

5) Let $(F,T)\ \theta_R(F,T)=(H,T\cap T)$. Thus, for every $\alpha\in T$, $H(\alpha)=F'(\alpha)\cap F'(\alpha)=F'(\alpha)$. Hence $(H,T)=(F,T)^r$. That is, the operation θ_R is not idempotent in $S_E(U)$.

6) Let $\emptyset_T=(S,T)$, where for every $\alpha\in T$, $S(\alpha)=\emptyset$. Let $(F,T)\theta_R(S,T)=(H,T\cap T)$, where for every $\alpha\in T$, $H(\alpha)=F'(\alpha)\cap S'(\alpha)=F'(\alpha)\cap U=F'(\alpha)$. Thus, $(H,T)=(F,T)^r$.

7) Let $\emptyset_M=(S,M)$, where for every $\alpha\in M$, $S(\alpha)=\emptyset$. Let $(S,M)\theta_R(F,T)=(H,M\cap T)$, where for every $\alpha\in T$, $H(\alpha)=S(\alpha)\cap F'(\alpha)=F'(\alpha)\cap U=F'(\alpha)$. Thus, $(H,T\cap M)=(F,T\cap M)^r$.

8) Let $\emptyset_E=(S,E)$, where for every $\alpha\in E$, $S(\alpha)=\emptyset$. $S(\alpha)=\emptyset$. Let $(F,T)\theta_R(S,E)=(H,T\cap E)$ where for every $\alpha\in T\cap E=T$, $H(\alpha)=F'(\alpha)\cap S'(\alpha)=F'(\alpha)\cap U=F'(\alpha)$.Thus, $(H,T)=(F,T)^r$.

9) Let $\emptyset_\emptyset=(S,\emptyset)$. Let $(F,T)\theta_R(S,\emptyset)=(H,T\cap\emptyset)$. Since the parameter set $\emptyset_\emptyset$ is the only soft set that is an empty set, $(H,\emptyset)=\emptyset_\emptyset$. That is, in the set $S_E(U)$, the absorbing element of the operation θ_R is the soft set $\emptyset_\emptyset$.

10) Let $U_T=(K,T)$, where for every $\alpha\in T$, $K(\alpha)=U$. Let $(F,T)\theta_R(K,T)=(H,T\cap T)$, where for every $\alpha\in T$, $H(\alpha)=F'(\alpha)\cap T'(\alpha)=F'(\alpha)\cap\emptyset=\emptyset$. Thus, $(H,T)=\emptyset_T$.

11) Let $U_M=(K,M)$, where for every $\alpha\in M$, $K(\alpha)=U$. Let $(F,T)\theta_R(K,M)=(H,T\cap M)$, where for every $\alpha\in T\cap M$, $H(\alpha)=F'(\alpha)\cap T'(\alpha)=F'(\alpha)\cap\emptyset=\emptyset$. Thus, $(H,T\cap M)=\emptyset_{T\cap M}$.

12) Let $U_E=(K,E)$, where for every $\alpha\in E$, $K(\alpha)=U$. Let $(F,T)\theta_R(K,E)=(H,T\cap E)$, where for every $\alpha\in T\cap E=T$, $H(\alpha)=F'(\alpha)\cap K'(\alpha)=F'(\alpha)\cap\emptyset=\emptyset$. Thus $(H,T)=\emptyset_T$.

13) Let $(F,T)^r=(H,T)$, where for every $\alpha\in T$, $H(\alpha)=F'(\alpha)$. Let $(F,T)\theta_R(H,T)=(L,T\cap T)$, where for every $\alpha\in T$, $L(\alpha)=F'(\alpha)\cap H'(\alpha)=F'(\alpha)\cap F(\alpha)=\emptyset$. Thus, $(L,T)=\emptyset_T$.

14) Let $(F,T)\theta_R(G,Z)=(H,T\cap Z)$, for every $\alpha\in T\cap Z$, $H(\alpha)=F'(\alpha)\cap G'(\alpha)$. Let $(H,T\cap Z)^r=(K,T\cap Z)$ where for every $\alpha\in T\cap Z$, $K(\alpha)=F(\alpha)\cup G(\alpha)$. Thus, $(K,T\cap Z)=(F,T)\cup_R(G,Z)$.

15) Let $(F,T)\theta_R(G,T)=(K,T\cap T)$, where for every $\alpha\in T$, $K(\alpha)=F'(\alpha)\cap G'(\alpha)$. Since $(K,T)=U_T$, $K(\alpha)=U$, for every $\alpha\in T$. Thus, $K(\alpha)=F'(\alpha)\cap G'(\alpha)=U$, for every $\alpha\in T\Leftrightarrow F'(\alpha)=U$ and $G'(\alpha)=U$, for every $\alpha\in T$ $\Leftrightarrow$ $F(\alpha)=\emptyset$ and $G(\alpha)=\emptyset$, for every $\alpha\in T\Leftrightarrow(F,T)=\emptyset_T$ and $(G,T)=\emptyset_T$, for every $\alpha\in T$.

16) Obvious.

17) Let $(F,T)\theta_R(G,Z)=(H,T\cap Z)$, where for every $\alpha\in T\cap Z$, $H(\alpha)=F'(\alpha)\cap G'(\alpha)$. Since, for every $\alpha\in T\cap Z$, $H(\alpha)=F'(\alpha)\cap G'(\alpha)\subseteq F'(\alpha)$.

Thus, $(F,T)\theta_R(G,Z)\tilde{\subseteq}(F,T)^r$. Similarly, since $F'(\alpha)\cap G'(\alpha)\subseteq G'(\alpha)$, $(F,T)\theta_R(G,Z)\tilde{\subseteq}(G,Z)^r$.

18) Let $(F,T)\tilde{\subseteq}(G,Z)$. Then, $T\subseteq Z$ and for every $\alpha\in T$, $F(\alpha)\subseteq G(\alpha)$. Thus for all $\alpha\in T$, $G'(\alpha)\subseteq F'(\alpha)$.

Let $(F,T)\theta_R(G,Z)=(K,T\cap Z=T)$. Then, for every $\alpha\in T$, $K(\alpha)=F'(\alpha)\cap G'(\alpha)=G'(\alpha)$, hence $(K,T)=(F,T)\theta_R(G,Z)=(G,T)^r$. Conversely let $(F,T)\theta_R(G,Z)=(G,T)^r$. Hence, $T\cap Z=T$, and so $T\subseteq Z$. Also, for every $\alpha\in T$, $F'(\alpha)\cap G'(\alpha)=G'(\alpha)$, and so $G'(\alpha)\subseteq F'(\alpha)$. Thus, for all $\alpha\in T$, $F(\alpha)\subseteq G(\alpha)$, $(F,T)\tilde{\subseteq}(G,Z)$.

19) Let $(F,T)\tilde{\subseteq}(G,T)$. Thus for every $\alpha\in T$, $F(\alpha)\subseteq G(\alpha)$ and for every $\alpha\in T$, $G'(\alpha)\subseteq F'(\alpha)$. Let $(G,T)\theta_R(H,Z)=(K,T\cap Z)$. Thus for every $\alpha\in T\cap Z$, $K(\alpha)=G'(\alpha)\cap H'(\alpha)$. Let $(F,T)\theta_R(H,Z)=(L,T\cap Z)$. Hence for every $\alpha\in T\cap Z$, $L(\alpha)=F'(\alpha)\cap H'(\alpha)$. Thus, $K(\alpha)=G'(\alpha)\cap H'(\alpha)\subseteq F'(\alpha)\cap H'(\alpha)=L(\alpha)$, for every $\alpha\in T\cap Z$, hence, $(G,T)\theta_R(H,Z)\tilde{\subseteq}(F,T)\theta_R(H,Z)$. It is clear from the commutative property that, under the same conditions, (H,Z) θ_R(G,T)$\widetilde{\subseteq}$(H,Z)θ_R(G,T) will be achieved.

20) We give a counterexample to show that the converse of Theorem 1 (19) is not true. Let E={e_1,e_2,e_3,e_4,e_5} be the parameter set, T={e_1,e_3}, K={e_1,e_3,e_5}, and Z={e_1,e_3,e_5,e_6} be the subsets of E, U={h_1,h_2,h_3,h_4,h_5} be the universal set, and (F,T), (G,T) and (H,Z) be the soft sets as follows: (F,T)={(e_1,{h_2,h_5,}),(e_3,{h_1,h_2,h_5,})},(G,T)={(e_1,{h_2}),(e_3,{h_1,h_2})},(H,Z)={(e_1,U),(e_3,U),(e_5,{h_1,h_5,})}.

Let (G,T) θ_R(H,Z)=(L,T∩ Z), where for every α ∈ T ∩ Z={e_1, e_3}, L(α)=G′(α) ∩H'(α), L(e_1)=G'(e_1)H'(e_1)=∅, L(e_3)=G'(e_3) ∩H'(e_3)= ∅. Thus, (G,T)θ_R(H,Z)={(e_1, ∅), (e_3, ∅)}. Now let (F,T)θ_R(H,Z)=(K,T∩Z), where for every α ∈ T ∩ Z={e_1, e_3}, K(α)=F'(α) ∩H'(α),K(e_1)=F'(e_1) ∩H'(e_1)=∅, K(e_3)=F'(e_3) ∩H'(e_3)= ∅. Thus, (F,T) θ_R(H,Z)={(e_1, ∅), (e_3, ∅)}.

It is observed that (G,T) θ_R(H,Z) $\widetilde{\subseteq}$(F,T) θ_R(H,Z); however then (F,T) $\widetilde{\subseteq}$ (G, K) needs not be true.

21) Let (F,T) $\widetilde{\subseteq}$ (G, T) and (K,V) $\widetilde{\subseteq}$ (L, V). Thus, for every α∈T and for every α∈Z, F(α)⊆ G(α) and K(α)⊆ L(α). Hence, for every α∈T, G'(α)⊆ F′(α) and for every α∈Z, L'(α)⊆ K′(α). Let (G,T) θ_R (L,Z)=(M,T∩ Z). Thus, for every α∈T∩ Z, M(α)=G'(α)∩L'(α). Let (F,T)θ_R (K,Z)=(N,T∩ Z). Thus, for every α∈T∩ Z, N(α)=F'(α)∩K'(α). Since, for every α∈T∩ Z, G'(α)⊆ F′(α) and L'(α)⊆ K′(α), M(α)=G'(α)∩L'(α)⊆ F′(α) ∩ K′(α) = N(α). Thus, (G,T) θ_R (L,V) $\widetilde{\subseteq}$(F,T) θ_R (K,V).

22) Let (F,T)θ_R(G,Z)=(M,T∩ Z). Hence, for every α∈T∩ Z, M(α)=F'(α)∩G'(α). Let (F,T)$*_R$(G,Z)=(N,T∩ Z). Thus, for every α∈T∩ Z, N(α)=F'(α)∪G'(α).

Since M(α)=F'(α)∩G'(α)⊆F'(α)∪G'(α)=N(α), it implies that (F,T)θ_R(G,Z)$\widetilde{\subseteq}$(F,T)$*_R$(G,Z).

Theorem 2. Let (F,T), (G,Z), and (H,M) be soft sets over U. Then, restricted theta operation distributes over other soft set operations as follows:

Theorem 3. Let (F,T), (G,Z), and (H,M) be soft sets over U. Then, restricted theta operation distributes over other restricted soft set operations as follows:

i) LHS Distributions:

1)(F,T)θ_R[(G,Z)$\cap_R$(H,M)]=[(F,T) θ_R(G,Z)]$\cup_R$ [(F,T) θ_R(H,M)].

Proof. Consider first the LHS. Let (G,Z)$\cap_R$(H,M)=(R,Z∩M), where for every α∈Z∩M, R(α)=G(α)∩H(α). Let (F,T) θ_R(R,Z∩M)=(N,T∩(Z∩M)), whee for every α∈T∩(Z∩M), N(α)= F'(α)∩R'(α). Thus, for every α∈T∩Z∩M,

$$N(\alpha)= F'(\alpha)\cap[(G'(\alpha)\cup H'(\alpha)]$$

Now consider the RHS, i.e. [(F,T) θ_R(G,Z)] $\cup_R$ [(F,T) θ_R(H,M)]. Let (F,T) θ_R (G,Z)=(V,T∩Z), where for every α∈T∩Z, V(α)=F'(α)∩G'(α) and let (F,T)θ_R (H,M)=(W,T∩M), where for every α∈T∩M, W(α)=F'(α)∩H'(α).

Let (V,T∩Z) $\cup_R$(W,T∩M)=(S,(T∩Z)∩(T∩M)), where for every α∈T∩Z∩M, S(α)=V(α)∪W(α). Thus,

$$S(\alpha)= [F'(\alpha)\cap G'(\alpha)]\cup[F'(\alpha)\cap H'(\alpha)]$$

Hence, (N,T∩Z∩M)=(S,T∩Z∩M). Here, if T∩Z=∅ or T∩M=∅ or Z∩M=∅, then both sides is $\emptyset_\emptyset$. Thus, the equality is satisfied in all circumstances.

2)(F,T) θ_R[(G,Z) $\cup_R$(H,M)]=[(F,T) θ_R(G,Z)]$\cap_R$[(F,T) θ_R (H,M)].

3)(F,T) θ_R [(G,Z) $*_R$(H,M)]=[(F,T) γ_R(G,Z)]$\cap_R$[(F,T) γ_R (H,M)].

4)(F,T) θ_R [(G,Z) θ_R(H,M)]=[(F,T) γ_R(G,Z)]$\cup_R$[(F,T) γ_R (H,M)].

ii) RHS Distributions:

1)[(F,T) $\cup_R$(G,Z)]θ_R(H,M)=[(F,T) θ_R(H,M)]$\cap_R$[(G,Z) θ_R (H M)].

Proof. Consider first the LHS. Let (F,T) $\cup_R$(G,Z)=(R,T∩Z), where for every α∈T∩Z, R(α)=F(α)∪G(α). Let (R,T∩Z) θ_R(H,M)=(N,(T∩Z)∩M)), where for every α∈(T∩Z)∩M, N(α)=R'(α)∩H'(α). Thus,

$$N(\alpha)= [F'(\alpha)\cap G'(\alpha)]\cap H'(\alpha)$$

Now consider the RHS, i.e., [(F,T) θ_R(H,M)] $\cap_R$ [(G,Z) θ_R (H,M)]. Let (F,T)θ_R(H,M)=(S,T∩M), where for every α∈T∩M, T(α)=F'(α)∩H'(α) and let (G,Z) θ_R (H,M)=(K,Z∩M), where for every α∈Z∩M, K(α)=G'(α)∩H'(α). Assume that (S,T∩Z) $\cap_R$(K,Z∩M)=(L,(T∩Z∩M)), where for every α∈(T∩Z)∩(Z∩M), L(α)=S(α)∩K(α). Thus,

$$L(\alpha)= ([F'(\alpha)\cap H'(\alpha)] \cap [G'(\alpha)\cap H'(\alpha)]$$

Hence, (N,T∩Z∩M)=(L,T∩Z∩M). Here, if T∩Z=∅ or T∩M=∅ or Z∩M=∅, then both sides is $\emptyset_\emptyset$. Thus, the equality is satisfied in all circumstances.

2)[(F,T)$\cap_R$ (G,Z)]θ_R(H,M)=[(F,T) θ_R(H,M)]$\cup_R$[(G,Z) θ_R(H,M)].

3)[(F,T) θ_R(G,Z)]θ_R (H,M)=[(F,T)$\backslash_R$(H,M)]$\cup_R$ [(G,Z)$\backslash_R$(H,M)].

4)[(F,T) $*_R$(G,Z)]θ_R(H,M)=[(F,T) $\backslash_R$ (H,M)]$\cap_R$ [(G,Z) $\backslash_R$ (H,M)].

Theorem 4. Let (F,T), (G,Z), and (H,M) be soft sets over U. Then, restricted theta operation distributes over extended soft set operations as follows:

i) LHS Distributions:

1)(F,T) θ_R[(G,Z) $\cap_\varepsilon$(H,M)]=[(F,T) θ_R(G,Z)] $\cap_\varepsilon$[(F,T) θ_R (H,M)].

Proof. Consider first the LHS. Let (G,Z) $\cap_\varepsilon$ (H,M)=(R,Z∪M), where for every α∈Z∪M;

$$R(\alpha) = \begin{cases} G(\alpha), & \alpha \in Z - M \\ H(\alpha), & \alpha \in M - Z \\ G(\alpha) \cap H(\alpha), & \alpha \in Z \cap M \end{cases}$$

Let $(F,T)\theta_R(R,Z\cup M)=(N,(T\cap(Z\cup M))$, where for every $\alpha\in T\cap(Z\cup M)$, $N(\alpha)=F'(\alpha)\cap R'(\alpha)$. Thus,

$$N(\alpha)=\begin{cases} F'(\alpha)\cap G'(\alpha), & \alpha\in T\cap(Z-M)\\ F'(\alpha)\cap H'(\alpha) & \alpha\in T\cap(M-Z)\\ F'(\alpha)\cap[G'(\alpha)\cup H'(\alpha)], & \alpha\in T\cap(Z\cap M)\end{cases}$$

Now consider the RHS, i.e. $[(F,T)\ \theta_R(G,Z)]\cap_\varepsilon [(F,T)\ \theta_R\ (H,M)]$. Let $(F,T)\theta_R(G,Z)=(K,T\cap Z)$, where for every $\alpha\in T\cap Z$, $K(\alpha)=F'(\alpha)\cap G'(\alpha)$ and let $(F,T)\theta_R(H,M)=(S,T\cap M)$, where for every $\alpha\in T\cap M$, $S(\alpha)=F'(\alpha)\cap H'(\alpha)$. Let $(K,T\cap Z)\cap_\varepsilon(S,T\cap M)=(L,(T\cap Z)\cup(T\cap M))$, where for every $\alpha\in(T\cap Z)\cup(T\cap M)$,

$$L(\alpha)=\begin{cases} K(\alpha), & \alpha\in(T\cap Z)-(T\cap M)\\ S(\alpha), & \alpha\in(T\cap Z)-(T\cap M)\\ K(\alpha)\cap S(\alpha), & \alpha\in(T\cap M)\cap(T\cap Z)\end{cases}$$

Thus,

$$L(\alpha)=\begin{cases} F'(\alpha)\cap G'(\alpha), & \alpha\in T\cap Z\cap M'\\ F'(\alpha)\cap H'(\alpha) & \alpha\in T\cap Z'\cap M\\ F'(\alpha)\cap[G'(\alpha)\cup H'(\alpha)], & \alpha\in T\cap Z\cap M\end{cases}$$

Hence, $(N,\ T\cap(Z\cup M))=(L,\ (T\cap Z)\cup(T\cap M))$. Here, if $T\cap Z=\emptyset$, then $N(\alpha)=L(\alpha)=F'(\alpha)\cap H'(\alpha)$; and if $T\cap M=\emptyset$, then $N(\alpha)=L(\alpha)=F'(\alpha)\cap G'(\alpha)$. Thus, there is no extra condition as $T\cap Z\neq\emptyset$ and/or $T\cap M\neq\emptyset$ for satisfying Theorem 4 (i).

2)$(F,T)\ \theta_R[(G,Z)\cup_\varepsilon(H,M)]=[(F,T)\ \theta_R(G,Z)]\cap_\varepsilon[(F,T)\ \theta_R\ (H,M)]$.

ii) RHS Distributions:

1)$[(F,T)\cup_\varepsilon(G,Z)]\ \theta_R(H,M)=[(F,T)\ \theta_R(H,M)]\cap_\varepsilon[(G,Z)\ \theta_R\ (H,M)]$.

Proof. Consider first the LHS. Let $(F,T)\cup_\varepsilon(G,Z)=(R,T\cup Z)$, where for every $\alpha\in T\cup Z$,

$$R(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-Z\\ G(\alpha), & \alpha\in Z-T\\ F(\alpha)\cup G(\alpha), & \alpha\in T\cap Z\end{cases}$$

Assume that $(R,T\cup Z)\ \theta_R(H,M)=(N,(T\cup Z)\cap M))$, where for every $\alpha\in(T\cup Z)\cap M$, $N(\alpha)=R'(\alpha)\cap H'(\alpha)$. Thus,

$$N(\alpha)=\begin{cases} F'(\alpha)\cap H'(\alpha), & \alpha\in(T-Z)\cap M\\ G'(\alpha)\cap H'(\alpha), & \alpha\in(Z-T)\cap M\\ [F'(\alpha)\cap G'(\alpha)]\cap H'(\alpha), & \alpha\in(T\cap Z)\cap M\end{cases}$$

Now consider the RHS. Let $(F,T)\ \theta_R\ (H,M)=(K,T\cap M)$, where for every $\alpha\in T\cap M$, $K(\alpha)=F'(\alpha)\cap H'(\alpha)$ and let $(G,Z)\ \theta_R\ (H,M)=(S,Z\cap M)$, where for every $\alpha\in Z\cap M$, $S(\alpha)=G'(\alpha)\cap H'(\alpha)$. Let $(K,T\cap M)\cap_\varepsilon(S,Z\cap M)=(L,(T\cap M)\cup(Z\cap M))$. Hence,

$$L(\alpha)=\begin{cases} K(\alpha), & \alpha\in(T\cap M)-(Z\cap M)\\ S(\alpha), & \alpha\in(Z\cap M)-(T\cap M)\\ K(\alpha)\cap S(\alpha), & \alpha\in(T\cap M)\cap(Z\cap M)\end{cases}$$

Thus,

$$L(\alpha)=\begin{cases} F'(\alpha)\cap H'(\alpha), & \alpha\in T\cap Z'\cap M\\ G'(\alpha)\cap H'(\alpha) & \alpha\in T'\cap Z\cap M\\ [F'(\alpha)\cap G'(\alpha)]\cap H'(\alpha), & \alpha\in T\cap Z\cap M\end{cases}$$

Therefore, $(N,(T\cup Z)\cap M))=(L,(T\cap M)\cup(Z\cap M))$. Here, if $T\cap Z=\emptyset$ and $\alpha\in T\cap Z'\cap M$, then $N(\alpha)=L(\alpha)=F'(\alpha)\cap H'(\alpha)$ and if $T\cap Z=\emptyset$ and $\alpha\in T'\cap Z\cap M$, the $N(\alpha)=L(\alpha)=G'(\alpha)\cap H'(\alpha)$. Furthermore, if $Z\cap M=\emptyset$, then $N(\alpha)=L(\alpha)=F'(\alpha)\cap H'(\alpha)$. Thus, there is no extra condition as $T\cap Z\neq\emptyset$ and/or $Z\cap M\neq\emptyset$ for satisfying Theorem 4 (ii).

2)$[(F,T)\cap_\varepsilon(G,Z)]\ \theta_R(H,M)=[(F,T)\ \theta_R(G,Z)]\cup_\varepsilon[(G,Z)\ \theta_R\ (H,M)]$.

Theorem 5. Let (F,T), (G,Z), and (H,M) be soft sets over U. Then, restricted theta operation distributes over complementary extended soft set operations as follows:

i) LHS Distributions:

1)$(F,T)\ \theta_R[(G,Z)\ \overset{*}{*}_\varepsilon\ (H,M)]=[(F,T)\ \gamma_R(G,Z)]\cap_\varepsilon[(F,T)\ \gamma_R\ (H,M)]$.

Proof. Consider first the LHS. Let $(G,Z)\overset{*}{*}_\varepsilon(H,M)=(R,Z\cup M)$, where for every $\alpha\in Z\cup M$,

$$R(\alpha)=\begin{cases} G'(\alpha), & \alpha\in Z-M\\ H(\alpha), & \alpha\in M-Z\\ G'(\alpha)\cup H'(\alpha), & \alpha\in Z\cap M\end{cases}$$

Let $(F,T)\theta_R(R,Z\cup M)=(N,(T\cap(Z\cup M))$, where for every $\alpha\in T\cap(Z\cup M)$, $N(\alpha)=F'(\alpha)\cap R'(\alpha)$. Thus,

$$N(\alpha)=\begin{cases} F'(\alpha)\cap G(\alpha), & \alpha\in T\cap(Z-M)\\ F'(\alpha)\cap H(\alpha) & \alpha\in T\cap(M-Z)\\ F'(\alpha)\cap[G(\alpha)\cap H(\alpha)], & \alpha\in T\cap(Z\cap M)\end{cases}$$

Now consider the RHS, i.e. $[(F,T)\ \gamma_R(G,Z)]\cap_\varepsilon[(F,T)\ \gamma_R\ (H,M)]$. Let $(F,T)\ \gamma_R\ (G,Z)=(K,T\cap Z)$ where for every $\alpha\in T\cap Z$, $K(\alpha)=F'(\alpha)\cap G(\alpha)$.

Let $(F,T)\gamma_R(H,M)=(S,T\cap M)$, where for every $\alpha\in T\cap M$, $S(\alpha)=F'(\alpha)\cap H(\alpha)$. Assume that $(K,T\cap Z)\cap_\varepsilon(S,T\cap M)=(L,(T\cap Z)\cup(T\cap M))$, where for every $\alpha\in(T\cap Z)\cup(T\cap M)$,

$$L(\alpha)=\begin{cases} K(\alpha), & \alpha\in(T\cap Z)-(T\cap M)\\ S(\alpha), & \alpha\in(T\cap Z)-(T\cap M)\\ K(\alpha)\cap S(\alpha), & \alpha\in(T\cap M)\cap(T\cap Z)\end{cases}$$

Thus,

$$L(\alpha)=\begin{cases} F'(\alpha)\cap G(\alpha), & \alpha\in T\cap Z\cap M'\\ F'(\alpha)\cap H(\alpha) & \alpha\in T\cap Z'\cap M\\ F'(\alpha)\cap[G(\alpha)\cup H(\alpha)], & \alpha\in T\cap Z\cap M\end{cases}$$

Therefore, $(N,(T\cap(Z\cup M))=(L,(T\cap Z)\cup(T\cap M))$. Here, if $T\cap Z=\emptyset$, then $N(\alpha)=L(\alpha)=F'(\alpha)\cap H(\alpha)$; and if $T\cap M=\emptyset$, then

N(α)=L(α)=F'(α)∩G(α). Thus, there is no extra condition as T∩Z≠ ∅ and/or T∩M≠ ∅ for satisfying Theorem 5 (i).

2)(F,T) θ_R [(G,Z) $\overset{*}{\theta}_{\varepsilon}$(H,M)]=[(F,T) γ_R(G,Z)] $\cup_R$[(F,T) γ_R (H, M)].

ii) RHS Distributions:

1)[(F,T) $\overset{*}{\theta}_{\varepsilon}$(G,Z)]$\theta_R$(H,M)=[(F,T) $\backslash_R$(H,M)]$\cup_\varepsilon$[(G,Z) $\backslash_R$ (H,M)].

Proof. Consider first the LHS. Let (F,T) $\overset{*}{\theta}_{\varepsilon}$ (G,Z)=(R,T∪Z), where for every α∈T∪Z;

$$R(\alpha)=\begin{cases} F'(\alpha), & \alpha\in T-Z\\ G'(\alpha), & \alpha\in Z-T\\ F'(\alpha)\cap G'(\alpha), & \alpha\in T\cap Z\end{cases}$$

Let (R,T∪Z) θ_R(H,M)=(N,(T∪Z)∩M), where for every α∈(T∪Z)∩M, N(α)=R'(α)∩H'(α). Thus,

$$N(\alpha)=\begin{cases} F(\alpha)\cap H'(\alpha), & \alpha\in(T-Z)\cap M\\ G(\alpha)\cap H'(\alpha) & \alpha\in(Z-T)\cap M\\ [F(\alpha)\cup G(\alpha)]\cap H'(\alpha), & \alpha\in(T\cap Z)\cap M\end{cases}$$

Now consider the RHS, i.e. [(F,T) $\backslash_R$(H,M)] $\cup_\varepsilon$[(G,Z) $\backslash_R$ (H,M)]. Let (F,T)$\backslash_R$(H,M)=(K,T∩M), where for every α∈T∩M, K(α)=F(α)∩H'(α) and let (G,Z) $\backslash_R$(H,M)=(S,Z∩M), where for every α∈Z∩M, S(α)=G(α)∩H'(α).

Assume that (K,T∩M)$\cup_\varepsilon$(S,Z∩M)=(L,(T∩M)∪(Z∩M)), where for every α∈(T∩M)∪(Z∩M),

$$L(\alpha)=\begin{cases} K(\alpha), & \alpha\in(T\cap M)-(Z\cap M)\\ S(\alpha), & \alpha\in(Z\cap M)-(T\cap M)\\ K(\alpha)\cup S(\alpha), & \alpha\in(T\cap M)\cap(Z\cap M)\end{cases}$$

Thus,

$$L(\alpha)=\begin{cases} F(\alpha)\cap H'(\alpha), & \alpha\in T\cap Z'\cap M\\ G(\alpha)\cap H'(\alpha) & \alpha\in T'\cap Z\cap M\\ [F(\alpha)\cup G(\alpha)]\cap H'(\alpha), & \alpha\in T\cap Z\cap M\end{cases}$$

Therefore, (N,(T∪Z)∩M)=(L,(T∩M)∪(Z∩M)). Here, if T∩Z=∅ and α∈T∩Z'∩M, then N(α)=L(α)=F(α)∩H'(α) and if T∩Z=∅ and α∈T'∩Z∩M, the N(α)=L(α)=G(α)∩H'(α). Furthermore, if Z∩M=∅, then N(α)=L(α)=F(α)∩H'(α). Thus, there is no extra condition as T∩Z≠ ∅ and/or Z∩M≠ ∅ for satisfying Theorem 5 (ii).

2)[(F,T) $\overset{*}{*}_{\varepsilon}$(G,Z)] θ_R(H,M)=[(F,T)$\backslash_R$(G,Z)]$\cap_\varepsilon$[(G,Z) $\backslash_R$ (H,M)].

Theorem 6. Let (F,T), (G,Z), and (H,M) be soft sets over U. Then, restricted theta operation distributes over soft binary piecewise operations as follows:

i) LHS Distributions:

1)(F,T) θ_R[(G,Z) $\tilde{\cap}$ (H,M)]=[(F,T) θ_R(G,Z)] $\tilde{\cup}$ [(F,T) θ_R (H,M)].

Proof. Consider first the LHS. Let (G,Z)$\tilde{\cap}$ (H,M)=(R,Z), where for every α∈Z;

$$R(\alpha)=\begin{cases} G(\alpha), & \alpha\in Z-M\\ G(\alpha)\cap H(\alpha), & \alpha\in Z\cap M\end{cases}$$

Let (F,T)θ_R(R,Z)=(N,T∩Z), where for every α∈T∩Z; N(α)=F'(α)∩R'(α). Thus,

$$N(\alpha)=\begin{cases} F'(\alpha)\cap G'(\alpha), & \alpha\in T\cap(Z-M)\\ F'(\alpha)\cap[G'(\alpha)\cup H'(\alpha)], & \alpha\in T\cap(Z\cap M)\end{cases}$$

Now consider the RHS. Let (F,T)θ_R(G,Z)=(K,T∩Z), where for every α∈T∩Z; K(α)=F'(α)∩G'(α).

Let (F,T) θ_R (H,M)=(S,T∩M), where for every α∈T∩M; S(α)=F'(α)∩H'(α) and assume that (K,T∩Z)$\tilde{\cup}$ (S,T∩M)=(L,T∩Z), where for every α∈T∩Z;

$$L(\alpha)=\begin{cases} K(\alpha), & \alpha\in(T\cap Z)-(T\cap M)\\ K(\alpha)\cup S(\alpha), & \alpha\in(T\cap Z)\cap(T\cap M)\end{cases}$$

Thus,

$$L(\alpha)=\begin{cases} F'(\alpha)\cap G'(\alpha), & \alpha\in(T\cap Z)-(T\cap M)\\ F'(\alpha)\cap\{G'(\alpha)\cup H'(\alpha)], & \alpha\in T\cap(Z\cap M)\end{cases}$$

Hence (N,T∩Z)=(L,T∩Z). Here, if T∩Z=∅, then (N,T∩Z)=(L,T∩Z)=$\emptyset_\emptyset$; and if T∩M=∅, then N(α)=L(α)=F'(α∩)G'(α). Thus, there is no extra condition as T∩Z≠ ∅ and/or T∩M≠ ∅ for satisfying Theorem 6 (i).

2)(F,T) θ_R[(G,Z) $\tilde{\cup}$(H,M)]=[(F,T)θ_R(G,Z)] $\tilde{\cap}$ [(F,T) θ_R (H,M)] .

ii) RHS Distributions:

1)[(F,T) $\tilde{\cup}$(G,Z)] θ_R(H,M)=[(F,T) θ_R(H,M)]$\tilde{\cap}$ [(G,Z) θ_R (H,M)].

Proof. Consider first the LHS. Let (F,T) $\tilde{\cup}$ (G,Z)=(R,T) , where for every α∈T;

$$R(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-Z\\ F(\alpha)\cup G(\alpha), & \alpha\in T\cap Z\end{cases}$$

Let (R,T)θ_R(H,M)=(N,T∩M), where for every α∈T∩M; N(α)=R'(α)∩H'(α). Thus,

$$N(\alpha)=\begin{cases} F'(\alpha)\cap H'(\alpha), & \alpha\in(T-Z)\cap M\\ [F'(\alpha)\cap G'(\alpha)]\cap H'(\alpha), & \alpha\in(T\cap Z)\cap M\end{cases}$$

Now consider the RHS. Let (F,T)θ_R(H,M)=(K,T∩M), where for every α∈T∩M; K(α)=F'(α)∩H'(α). Assume that (G,Z) θ_R (H,M)=(S,Z∩M), where for every α∈Z∩M; S(α)=G'(α)∩H'(α) and let (K,T∩M) $\tilde{\cap}$(S,Z∩M)=(L,T∩M), where for every α∈T∩M;

$$L(\alpha)=\begin{cases} K(\alpha), & \alpha \in (T\cap M)-(Z\cap M)\\ K(\alpha)\cap S(\alpha), & \alpha \in (T\cap M)\cap(Z\cap M)\end{cases}$$

Hence,

$$L(\alpha)=\begin{cases} F'(\alpha)\cap H'(\alpha), & \alpha \in (T\cap M)-(Z\cap M)\\ [F'(\alpha)\cap G'(\alpha)]\cap H'(\alpha), & \alpha \in (T\cap Z)\cap M\end{cases}$$

Thus, $(N,T\cap M)=(L,T\cap M)$. Here, if $T\cap M=\emptyset$, then $(N,T\cap M)=(L,T\cap M)=\emptyset_{\emptyset}$; and if $Z\cap M=\emptyset$, then $N(\alpha)=L(\alpha)=F'(\alpha)\cap H'(\alpha)$. Thus, there is no extra condition as $T\cap M\neq\emptyset$ and/or $Z\cap M\neq\emptyset$ for satisfying Theorem 6 (ii).

2)$[(F,T)\,\tilde{\cap}\,(G,Z)]\ \theta_R(H,M)=[(F,T)\theta_R(H,M)]\,\tilde{\cup}\,[(G,Z)\ \theta_R\ (H,M)]$.

Extended Theta Operation and Its Properties

Definition 14. Let (F,T) and (G,Z) be soft sets over U. The extended theta operation of (F,T) and (G,Z) is the soft set (H,C), denoted by $(F,T)\ \theta_\varepsilon\ (G,Z)=(H,C)$, where $C=T\cup Z$ and for every $\alpha\in C$,

$$H(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-Z\\ G(\alpha), & \alpha\in Z-T\\ F'(\alpha)\cap G'(\alpha), & \alpha\in T\cap Z\end{cases}$$

From the definition, it is obvious that if $T=\emptyset$, then $(F,T)\theta_\varepsilon(G,Z)=(G,Z)$; if $Z=\emptyset$, then $(F,T)\theta_\varepsilon(G,Z)=(F,T)$; if $T=Z=\emptyset$, then $(F,T)\theta_\varepsilon(G,Z)=\emptyset_{\emptyset}$.

Example 2. Let $E=\{e_1,e_2,e_3,e_4\}$be the parameter set, $T=\{e_1,e_3\}$ and $Z=\{e_2,e_3,e_4\}$ be subsets of E, $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the universal set, (F,T) and (G,Z) be the soft sets over U as $(F,T)=\{(e_1,\{h_2,h_5\}),(e_3,\{h_1,h_2,h_5\})\}$,$(G,Z)=\{(e_2,\{h_1,h_4,h_5\}),\{(e_3,\{h_2,h_3,h_4\}),(e_4,\{h_3,h_5\})\}$. Here let $(F,T)\ \theta_\varepsilon(G,Z)=(H,T\cup Z)$, where for every $\alpha\in T\cup Z$;

$$H(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-Z\\ G(\alpha), & \alpha\in Z-T\\ F'(\alpha)\cap G'(\alpha), & \alpha\in T\cap Z\end{cases}$$

Since $T-Z=\{e_1\}$, $Z-T=\{e_2,e_4\}$, $T\cap Z=\{e_3\}$, thus, $H(e_1)=F(e_1)=\{h_2,h_5\}$,$H(e_2)=G(e_2)=\{h_1,h_4,h_5\}$,$H(e_4)=G(e_4)=\{h_3,h_5\}$, $H(e_3)=F'(e_3)\cap G'(e_3)=\{h_3,h_4\}\cap\{h_1,h_5\}=\emptyset$. Thus, $(F,T)\theta_\varepsilon(G,Z)=\{(e_1,\{h_2,h_5\}),(e_2,\{h_1,h_4,h_5\}),(e_3,\emptyset),(e_4,\{h_3,h_5\})\}$

Remark 1. In the set $S_T(U)$, where T is a fixed subset of E, restricted and extended theta operations coincide with each other. That is, $(F,T)\ \theta_\varepsilon(G,T)=(F,T)\ \theta_R(G,T)$.

Theorem 7. Let (F,T), (G,Z), (H,M), (G,T), (H,T), (K,T) and (L,T), be soft sets over U. Then, we have the followings:

1) The set $S_E(U)$ and $S_T(U)$ are closed under θ_ε.
2) If $T\cap Z\cap M=\emptyset$, then $[(F,T)\theta_\varepsilon(G,Z)]\ \theta_\varepsilon(H,M)=(F,T)\ \theta_\varepsilon[(G,Z)\ \theta_\varepsilon\ (H,M)]$.
3) $[(F,T)\ \theta_\varepsilon(G,T)]\ \theta_\varepsilon\ (H,T)\neq(F,T)\ \theta_\varepsilon[(G,T)\ \theta_\varepsilon(H,T)]$.
4) $(F,T)\ \theta_\varepsilon(G,Z)=(G,Z)\ \theta_\varepsilon\ (F,T)$.
5) $(F,T)\ \theta_\varepsilon(F,T)=(F,T)^r$.
6) $(F,T)\ \theta_\varepsilon\emptyset_T=\emptyset_T\theta_\varepsilon\ (F,T)=U_T$.
7) $(F,T)\theta_\varepsilon\emptyset_\emptyset=(F,T)$.
8) $\emptyset_\emptyset\ \theta_\varepsilon(F,T)=(F,T)$.
9) $(F,T)\ \theta_\varepsilon U_T=U_T\theta_\varepsilon(F,T)=\emptyset_T$.
10) $(F,T)\ \theta_\varepsilon(F,T)^r=(F,T)^r\theta_\varepsilon\ (F,T)=\emptyset_T$.
11) $[(F,T)\ \theta_\varepsilon(G,Z)]^r=(F,T)\ \underset{\cup}{\overset{*}{\sim}}\ (G,Z)$.
12) $(F,T)\ \theta_\varepsilon(G,T)=U_T$ if and only if $(F,T)=\emptyset_T$ and $(G,T)=\emptyset_T$.
13) $\emptyset_T\ \tilde{\subseteq}(F,T)\theta_\varepsilon(G,Z)$, $\emptyset_Z\ \tilde{\subseteq}(F,T)\theta_\varepsilon(G,Z)$. Moreover, $(F,T)\theta_\varepsilon\ (G,Z)\ \tilde{\subseteq}\ U_{T\cup Z}$.
14) $(F,T)\theta_\varepsilon(G,T)\ \tilde{\subseteq}\ (F,T)^r$ and $(F,T)\theta_\varepsilon(G,T)\ \tilde{\subseteq}\ (G,T)^r$.
15) If $(F,T)\ \tilde{\subseteq}\ (G,T)$, $(F,T)\theta_\varepsilon(G,T)=(G,T)^r$.
16) If $(F,T)\tilde{\subseteq}(G,T)$, $(G,T)\theta_\varepsilon(H,T)\tilde{\subseteq}(F,T)\theta_\varepsilon(H,T)$. The converse is not true.
17) If $(F,T)\ \tilde{\subseteq}\ (G,T)$ and $(K,T)\ \tilde{\subseteq}\ (L,T)$, $(G,T)\ \theta_\varepsilon(L,T)\ \tilde{\subseteq}(F,T)\ \theta_\varepsilon(K,T)$.
18) $(F,T)\ \theta_\varepsilon\ (G,Z)\ \tilde{\subseteq}(F,T)*_\varepsilon(G,Z)$ and $(G,Z)\ \theta_\varepsilon(F,T)\ \tilde{\subseteq}(G,Z)*_\varepsilon(F,T)$.

Proof. 1) It is clear that θ_ε is a binary operation in $S_E(U)$. That is,

$$\theta_\varepsilon: S_E(U)\times S_E(U)\rightarrow S_E(U)$$
$$((F,T),(G,Z))\rightarrow(F,T)\theta_\varepsilon(G,Z)=(H,T\cup Z)$$

Namely, when (F,T) and (G,Z) are soft set over U, then so $(F,T)\ \theta_\varepsilon\ (G,Z)$. Similarly, $S_T(U)$ is closed under θ_ε. That is,

$$\theta_\varepsilon: S_T(U)\times S_T(U)\rightarrow S_T(U)$$
$$((F,T),(G,T))\rightarrow(F,T)\theta_\varepsilon(G,T)=(K,T\cup T)=(K,T)$$

Namely, θ_ε is a binary operation in $S_T(U)$.

2) First, consider the LHS. Let $(F,T)\theta_\varepsilon(G,Z)=(S,T\cup Z)$, where for every $\alpha\in T\cup Z$,

$$S(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-Z\\ G(\alpha), & \alpha\in Z-T\\ F'(\alpha)\cap G'(\alpha), & \alpha\in T\cap Z\end{cases}$$

Let $(S,T\cup Z)\theta_\varepsilon(H,M)=(N,(T\cup Z)\cup M))$, where for every $\alpha\in(T\cup Z)\cup M$,

$$N(\alpha)=\begin{cases} S(\alpha), & \alpha\in(T\cup Z)-M\\ H(\alpha), & \alpha\in M-(T\cup Z)\\ S'(\alpha)\cap H'(\alpha), & \alpha\in(T\cup Z)\cap M\end{cases}$$

Thus,

$$N(\alpha)=\begin{cases} F(\alpha), & \alpha\in(T-Z)-M\\ G(\alpha) & \alpha\in(Z-T)-M\\ F'(\alpha)\cap G'(\alpha) & \alpha\in(T\cap Z)-M\\ H(\alpha), & \alpha\in M\text{-}(T\cup Z)\\ F'(\alpha)\cap H'(\alpha), & \alpha\in(T-Z)\cap M\\ G'(\alpha)\cap H'(\alpha), & \alpha\in(Z-T)\cap M\\ [F(\alpha)\cup G(\alpha)]\cap H'(\alpha) & \alpha\in(T\cap Z)\cap M\end{cases}$$

Now consider the RHS. Let $(G,Z)\ \theta_\varepsilon(H,M)=(R,Z\cup M)$, where for every $\alpha\in Z\cup M$;

$$R(\alpha)=\begin{cases} G(\alpha), & \alpha\in Z-M\\ H(\alpha), & \alpha\in M-Z\\ G'(\alpha)\cap H'(\alpha), & \alpha\in Z\cap M\end{cases}$$

Let $(F,T)\theta_\varepsilon(R,Z\cup M)=(L,(T\cup(Z\cup M))$, where for every $\alpha\in T\cup Z\cup M$;

$$L(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-(Z\cup M)\\ R(\alpha), & \alpha\in (Z\cup M)-T\\ F'(\alpha)\cap R'(\alpha), & \alpha\in T\cap (Z\cup M)\end{cases}$$

Hence,

$$L(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-(Z\cup M)\\ G(\alpha) & \alpha\in (Z-M)-T\\ H(\alpha) & \alpha\in (M-Z)-T\\ G'(\alpha)\cap H'(\alpha) & \alpha\in (Z\cap M)-T\\ F'(\alpha)\cap G'(\alpha) & \alpha\in T\cap(Z-M)\\ F'(\alpha)\cap H'(\alpha) & \alpha\in T\cap(M-Z)\\ F'(\alpha)\cap\ [G(\alpha)\cup H(\alpha)] & \alpha\in T\cap(Z\cap M)\end{cases}$$

It is observed that $(N,(T\cup Z)\cup M)=(L,T\cup(Z\cup M))$, where $T\cap Z\cap M=\emptyset$. That is, in $S_E(U)$, θ_ε is associative under certain conditions.

3) The proof follows from Remark 1 and Theorem 1 (3). That is, in $S_T(U)$, where T is a fixed subset of E, θ_ε is not associative.

4) Let $(F,T)\ \theta_\varepsilon(G,Z)=(H,T\cup Z)$, where for every $\alpha\in T\cup Z$,

$$H(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-Z\\ G(\alpha), & \alpha\in Z-T\\ F'(\alpha)\cap G'(\alpha), & \alpha\in T\cap Z\end{cases}$$

Let $(G,Z)\ \theta_\varepsilon(F,T)=(S,Z\cup T)$, where for every $\alpha\in Z\cup T$,

$$S(\alpha)=\begin{cases} G(\alpha), & \alpha\in Z-T\\ F(\alpha), & \alpha\in T-Z\\ G'(\alpha)\cap F'(\alpha), & \alpha\in Z\cap T\end{cases}$$

Thus, $(F,T)\ \theta_\varepsilon\ (G,Z)=(G,Z)\ \ \theta_\varepsilon(F,T)$. Moreover, it is obvious that $(F,T)\ \theta_\varepsilon\ (G,T)=(G,T)\theta_\varepsilon(F,T)$. That is, in $S_E(U)$ and $S_T(U)$, θ_ε is commutative.

5) The proof follows from Remark 1 and Theorem 1 (5). That is, in $S_E(U)$, θ_ε is not idempotent.

6) The proof follows from Remark 1 and Theorem 1 (6).

7) Let $\emptyset_\emptyset=(S,\emptyset)$ and $(F,T)\ \theta_\varepsilon(S,\emptyset)=(H,T\cup\emptyset)$, where for every $\alpha\in T\cup\emptyset=T$,

$$H(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-\emptyset=T\\ S(\alpha), & \alpha\in\emptyset\ -T=\emptyset\\ F'(\alpha)\cap S'(\alpha), & \alpha\in T\cap\emptyset=\emptyset\end{cases}$$

Thus, $H(\alpha)=F(\alpha)$, for every $\alpha\in T$, implying that $(H,T)=(F,T)$.

8) Let $\emptyset_\emptyset=(S,\emptyset)$ and $(F,T)\ \theta_\varepsilon(S,\emptyset)=(H,T\cup\emptyset)$, where for every $\alpha\in T\cup\emptyset=T$,

$$H(\alpha)=\begin{cases} S(\alpha), & \alpha\in\emptyset-T=\emptyset\\ F(\alpha), & \alpha\in T-\emptyset=T\\ S'(\alpha)\cap F'(\alpha), & \alpha\in\emptyset\cap T=\emptyset\end{cases}$$

Thus, for every $\alpha\in T$, $H(\alpha)=F(\alpha)$, $(H,T)=(F,T)$.

By Theorem 7 (7) and (8), we can conclude that in $S_E(U)$, the identity element of θ_ε is the soft set $\emptyset_\emptyset$. In classical set theory, it is well-known that $A\cup B=\emptyset\Leftrightarrow A=\emptyset$ and $B=\emptyset$. Thus, it is evident that in $S_E(U)$, we cannot find $(G,K)\in S_E(U)$ such that $(F,T)\ \theta_\varepsilon(G,K)=(G,K)\ \theta_\varepsilon(F,T)=\emptyset_\emptyset$; as this situation requires that $T\cup K=\emptyset$ and thus, $T=\emptyset$ and $K=\emptyset$. Since in $S_E(U)$, the only soft set with an empty parameter set is $\emptyset_\emptyset$, it follows that only the identity element $\emptyset_\emptyset$ has an inverse and its inverse it its own as usual. Thus, in $S_E(U)$, any other element except $\emptyset_\emptyset$ does not have an inverse for the operation θ_ε.

Corollary 1. Let (F,T), (G,Z), and (H,M) be the elements of $S_E(U)$. By Theorem 7 (1), (2), (4), (7) and (8), $(S_E(U),\theta_\varepsilon)$ is a commutative monoid whose identity is $\emptyset_\emptyset$ where $T\cap Z\cap M=\emptyset$. Since $(S_A(U),\theta_\varepsilon)$ is not associative, where A is a fixed subset of E, this algebraic structure can not be a semigroup.

9) The proof follows from Remark 1 and Theorem 1 (10).

10) The proof follows from Remark 1 and Theorem 1 (13).

11) Let $(F,T)\ \theta_\varepsilon(G,Z)=(H,T\cup Z)$, where for every $\alpha\in T\cup Z$;

$$H(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-Z\\ G(\alpha), & \alpha\in Z-T\\ F'(\alpha)\cap G'(\alpha), & \alpha\in T\cap Z\end{cases}$$

Let $(H,T\cup Z)^r=(K,T\cup Z)$, for every $\alpha\in T\cup Z$;

$$K(\alpha)=\begin{cases} F'(\alpha), & \alpha\in T-Z\\ G'(\alpha), & \alpha\in Z-T\\ F(\alpha)\cup G(\alpha), & \alpha\in T\cap Z\end{cases}$$

Thus, $(K,T\cup Z)=(F,T)\ \overset{*}{\underset{\cup}{\sim}}\ (G,Z)$.

12) The proof follows from Remark 1 and Theorem 1 (15).

13) The proof is obvious.

14) The proof follows from Remark 1 and Theorem 1 (17).

15) The proof follows from Remark 1 and Theorem 1 (18).

16) The proof follows from Remark 1 and Theorem 1 (19) and (20).

17) The proof follows from Remark 1 and Theorem 1 (21).

18) Let $(F,T)\theta_\varepsilon(G,Z)=(H,TUZ)$, where for every $\alpha\in TUZ$,

$$H(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-Z\\ G(\alpha), & \alpha\in Z-T\\ F'(\alpha)\cap G'(\alpha), & \alpha\in T\cap Z\end{cases}$$

Let $(F,T)\ *_\varepsilon\ (G,Z)=(K,TUZ)$, where for every $\alpha\in TUZ$,

$$K(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-Z\\ G(\alpha), & \alpha\in Z-T\\ F'(\alpha)\cup G'(\alpha), & \alpha\in T\cap Z\end{cases}$$

for every $\alpha\in T-Z$; $H(\alpha)=F(\alpha)\subseteq F(\alpha)=K(\alpha)$, for every $\alpha\in Z-T$; $H(\alpha)=G(\alpha)\subseteq G(\alpha)=K(\alpha)$, for every $\alpha\in T\cap Z$; $H(\alpha)=F'(\alpha)\cap G'(\alpha)\subseteq F'(\alpha)\cup G'(\alpha)= H(\alpha)$, $(F,A)\theta_{\varepsilon}(G,B) \cong (F,A) *_{\varepsilon} (G,B)$ is obtained.

Theorem 8. Let (F,T), (G,Z), and (H,M) be soft sets over U. Then, extended theta operation distributes over other soft set operations as follows:

Theorem 9. Let (F,T), (G,Z), and (H,M) be soft sets over U. Then, extended theta operation distributes over extended soft set operations as follows:

i) LHS Distributions

The following equations are satisfied if $T\cap (Z\Delta M)=TZ \cap M=\emptyset$.

1) $(F,T)\ \theta_{\varepsilon}[(G,Z)\cup_{\varepsilon}(H,M)]=[(F,T)\ \theta_{\varepsilon}(G,Z)]\cup_{\varepsilon}[(F,T)\ \theta_{\varepsilon}(H,M)$.

Proof. Consider first the LHS. Let $(G,Z)\cup_{\varepsilon}(H,M)=(R,Z\cup M)$, where for every $\alpha\in Z\cup M$,

$$R(\alpha)=\begin{cases} G(\alpha), & \alpha\in Z-M \\ H(\alpha), & \alpha\in M-Z \\ G(\alpha)\cup H(\alpha), & \alpha\in Z\cap M \end{cases}$$

Let $(F,T)\ \theta_{\varepsilon}(R,Z\cup M)=(N,(T\cup(Z\cup M))$, where for every $\alpha\in T\cup(Z\cup M)$;

$$N(\alpha)=\begin{cases} F(\alpha) & \alpha\in T-(Z\cup M) \\ R(\alpha) & \alpha\in (Z\cup M)-T \\ F'(\alpha)\cap R'(\alpha) & \alpha\in T\cap(Z\cup M) \end{cases}$$

Thus,

$$N(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-(Z\cup M) \\ G(\alpha), & \alpha\in(Z-M)-T \\ H(\alpha), & \alpha\in (M-Z)-T \\ G(\alpha)\cup H(\alpha), & \alpha\in(T\cap Z)-T \\ F'(\alpha)\cap G'(\alpha), & \alpha\in T\cap(Z-M) \\ F'(\alpha)\cap H'(\alpha), & \alpha\in T\cap(M-Z) \\ F'(\alpha)\cap\ [G'(\alpha)\cap H'(\alpha)], & \alpha\in(T\cap Z)\cap M \end{cases}$$

Now consider the RHS, i.e.$[(F,T)\ \theta_{\varepsilon}(G,Z)]\cup_{\varepsilon}[(F,T)\ \theta_{\varepsilon}(H,M)]$. $(F,T)\theta_{\varepsilon}(G,Z)=(K,T\cup Z)$, where for every $\alpha\in T\cup Z$;

$$K(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-Z \\ G(\alpha), & \alpha\in Z-T \\ F'(\alpha)\cap G'(\alpha), & \alpha\in T\cap Z \end{cases}$$

Let $(F,T)\ \theta_{\varepsilon}(H,M)=(S,T\cup M)$, where for every $\alpha\in T\cup M$;

$$S(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-M \\ H(\alpha), & \alpha\in M-T \\ F'(\alpha)\cap H'(\alpha), & \alpha\in T\cap M \end{cases}$$

Assume that $(K,T\cup Z)\cup_{\varepsilon}(S,T\cup M)=(L,(T\cup Z)\cup(T\cup M))$, where for every $\alpha\in(T\cup Z)\cup(T\cup M)$ Thus,

$$L(\alpha)=\begin{cases} K(\alpha), & \alpha\in(T\cup Z)-(T\cup M) \\ S(\alpha), & \alpha\in(T\cup M)-(T\cup Z) \\ K(\alpha)\cup S(\alpha), & \alpha\in(T\cup M)-(T\cup Z) \end{cases}$$

Thus,

$$L(\alpha)=\begin{cases} F(\alpha), & \alpha\in(T-Z)-(T\cup M) \\ G(\alpha), & \alpha\in(Z-T)\text{-}(T\cup M) \\ F'(\alpha)\cap G'(\alpha), & \alpha\in(T\cap Z)-(T\cup M) \\ F(\alpha), & \alpha\in(T-M)-(T\cup Z) \\ H(\alpha), & \alpha\in(M-T)-(T\cup Z)=\emptyset \\ F'(\alpha)\cap H'(\alpha), & \alpha\in(T\cap M)-(T\cup Z) \\ F(\alpha)\cup F(\alpha), & \alpha\in(T-Z)\cap(T-M) \\ F(\alpha)\cup H(\alpha), & \alpha\in(T-Z)\cap(M-T)=\emptyset \\ F(\alpha)\cup[F'(\alpha)\cap H'(\alpha)], & \alpha\in(T-Z)\cap(T\cap M)=\emptyset \\ G(\alpha)\cup F(\alpha), & \alpha\in(Z-T)\cap(T-M)=\emptyset \\ G(\alpha)\cup H(\alpha), & \alpha\in(Z-T)\cap(M-T) \\ G(\alpha)\cup[F'(\alpha)\cap H'(\alpha)] & \alpha\in(Z-T)\cap(T\cap M) \\ [F'(\alpha)\cap G'(\alpha)]\cup F(\alpha), & \alpha\in(T\cap Z)\cap(T-M)=\emptyset \\ [F'(\alpha)\cap G'(\alpha)]\cup H(\alpha) & \alpha\in(T\cap Z)\cap(M-T) \\ [F'(\alpha)\cap G'(\alpha)]\cup\ [F'(\alpha)\cap H'(\alpha)], & \alpha\in T\cap Z\cap M \end{cases}$$

Hence,

$$L(\alpha)=\begin{cases} G(\alpha), & \alpha\in T'\cap Z\cap M \\ H(\alpha), & \alpha\in T'\cap Z'\cap M \\ F(\alpha), & \alpha\in T\cap Z'\cap M' \\ F(\alpha)\cup H'(\alpha), & \alpha\in T\cap Z'\cap M \\ G(\alpha)\cup H(\alpha), & \alpha\in T'\cap Z\cap M \\ G'(\alpha)\cup H(\alpha), & \alpha\in T\cap Z\cap M' \\ F'(\alpha)\cap[G'(\alpha)\cup H'(\alpha)], & \alpha\in T\cap Z\cap M \end{cases}$$

Therefore, N=L under the condition $T\cap Z'\cap M=T\cap Z\cap M'=T\cap Z\cap M=\emptyset$. It is obvious that the condition $T\cap Z'\cap M=T\cap Z\cap M'=\emptyset$ is equivalent to the condition $T\cap (Z\Delta M)=\emptyset$.

2) $(F,T)\ \theta_{\varepsilon}[(G,Z)\cap_{\varepsilon}(H,M)]=[(F,T)\ \theta_{\varepsilon}(G,Z)]\cup_{\varepsilon}[(F,T)\ \theta_{\varepsilon}(H,M)]$

ii) RHS Distributions

The following equations are satisfied if $(T\triangle Z)\cap M=T\cap Z\cap M=\emptyset$.

1) $[(F,T)\cap_{\varepsilon}(G,Z)]\ \theta_{\varepsilon}(H,M)=[(F,T)\ \theta_{\varepsilon}(H,M)]\cap_{\varepsilon}[(G,Z)\ \theta_{\varepsilon}(H,M)]$.

Proof. Consider first the LHS. Let $(F,T)\cap_{\varepsilon}(G,Z)=(R,T\cup Z$, where for every $\alpha\in T\cup Z$;

$$R(\alpha)=\begin{cases} F(\alpha), & \alpha\in T-Z \\ G(\alpha), & \alpha\in Z-T \\ F(\alpha)\cap G(\alpha), & \alpha\in T\cap Z \end{cases}$$

Let $(R,T\cup Z)\theta_{\varepsilon}(H,M)=(N,(T\cup Z)\cup M)$.Thus, for every $\alpha\in(T\cup Z)\cup M$;

$$N(\alpha)=\begin{cases} R(\alpha), & \alpha\in(T\cup Z)-M \\ H(\alpha), & \alpha\in M-(T\cup Z) \\ R'(\alpha)\cap H'(\alpha), & \alpha\in(T\cup Z)\cap M \end{cases}$$

Hence,

$$N(\alpha)=\begin{cases} F(\alpha), & \alpha \in (T-Z)-M \\ G(\alpha), & \alpha \in (Z-T)-M \\ H(\alpha), & \alpha \in M-(T\cup Z) \\ F(\alpha)\cap G(\alpha), & \alpha \in (T\cap Z)-M \\ F'(\alpha)\cap H'(\alpha), & \alpha \in (T-Z)\cap M \\ G'(\alpha)\cap H'(\alpha), & \alpha \in (Z-T)\cap M \\ [F'(\alpha)\cup G'(\alpha)]\cap H'(\alpha), & \alpha \in T\cap(Z\cap M) \end{cases}$$

Now consider the RHS. Let $(F,T)\theta_\varepsilon(H,M)=(S,T\cup M)$, where for every $\alpha\in T\cup M$;

$$S(\alpha)=\begin{cases} F(\alpha), & \alpha \in T-M \\ H(\alpha), & \alpha \in M-T \\ F'(\alpha)\cap H'(\alpha), & \alpha \in T\cap M \end{cases}$$

Let $(G,Z)\ \theta_\varepsilon\ (H,M)=(K,Z\cup M)$, where for every $\alpha\in Z\cup M$

$$K(\alpha)=\begin{cases} G(\alpha), & \alpha \in Z-M \\ H(\alpha), & \alpha \in M-Z \\ G'(\alpha)\cup H'(\alpha), & \alpha \in Z\cap M \end{cases}$$

Assume that $(S,T\cup Z)\cap_\varepsilon(K,Z\cup M)=(W,(T\cup Z)\cap(Z\cup M))$, where for every $\alpha\in(T\cup Z)\cup(Z\cup M)$;

$$L(\alpha)=\begin{cases} K(\alpha), & \alpha \in (T\cup Z)-(T\cup M) \\ S(\alpha), & \alpha \in (T\cup M)-(T\cup Z) \\ K(\alpha)\cap S(\alpha), & \alpha \in (T\cup M)-(T\cup Z) \end{cases}$$

Thus,

$$L(\alpha)=\begin{cases} F(\alpha), & \alpha\in(T-M)-(Z\cup M) \\ H(\alpha), & \alpha\in(M-T)-(Z\cup M) \\ F'(\alpha)\cap H'(\alpha), & \alpha\in(T\cap M)-(Z\cup M) \\ G(\alpha), & \alpha\in(Z-M)-(T\cup M) \\ H(\alpha), & \alpha\in(M-Z)-T\cup M=\emptyset \\ G'(\alpha)\cap H'(\alpha), & \alpha\in(Z\cap M)-(T\cup M) \\ F(\alpha)\cap G(\alpha), & \alpha\in(T-M)\cap(Z-M) \\ F(\alpha)\cap H(\alpha), & \alpha\in(T-M)\cap(M-Z)=\emptyset \\ F(\alpha)\cap[G'(\alpha)\cap H'(\alpha)], & \alpha\in(T-M)\cap(Z\cap M)=\emptyset \\ H(\alpha)\cap G(\alpha), & \alpha\in(M-T)\cap(Z-M)=\emptyset \\ H(\alpha)\cap H(\alpha), & \alpha\in(M-T)\cap(M-Z) \\ H(\alpha)\cap[G'(\alpha)\cap H'(\alpha)] & \alpha\in(M-T)\cap(Z\cap M) \\ H(\alpha)\cap[G'(\alpha)\cap H'(\alpha)], & \alpha\in(T\cap M)\cap(Z-M)=\emptyset \\ [F'(\alpha)\cap H'(\alpha)]\cap H(\alpha), & \alpha\in(T\cap M)\cap(M-Z) \\ [F'(\alpha)\cap H'(\alpha)]\cap[G'(\alpha)\cap H'(\alpha)], & \alpha\in T\cap Z\cap M \end{cases}$$

Hence,

$$L(\alpha)=\begin{cases} F(\alpha), & \alpha \in T\cap Z'\cap M' \\ G(\alpha), & \alpha \in T'\cap Z\cap M' \\ H(\alpha), & \alpha \in T'\cap Z'\cap M \\ F(\alpha)\cap G(\alpha), & \alpha \in T\cap Z\cap M' \\ \emptyset, & \alpha \in T'\cap Z\cap M \\ \emptyset, & \alpha \in T\cap Z'\cap M \\ [F'(\alpha)\cap G'(\alpha)\cap H'(\alpha), & \alpha \in T\cap Z\cap M \end{cases}$$

Therefore, N=L under the condition $T'\cap Z\cap M=T\cap Z'\cap M=T\cap Z\cap M'=\emptyset$. It is obvious that the condition $T'\cap Z\cap M=T\cap Z'\cap M=\emptyset$ is equivalent to the condition $(T\Delta Z)\cap M=\emptyset$.

2)$[(F,T)\cup_\varepsilon(G,Z)]\ \theta_\varepsilon(H,M)=[(F,T)\ \theta_\varepsilon(H,M)]\cap_\varepsilon[(G,Z)\ \theta_\varepsilon\ (H,M)]$.

Corollary 2. $(S_E(U),\cup_\varepsilon,\theta_\varepsilon)$ is an additive idempotent commutative semiring without zero, but with unity under certain conditions.

Proof. Ali et al. (2011) showed that $(S_E(U),\cup_\varepsilon)$ is a commutative, idempotent monoid with identity $\emptyset_\emptyset$, that is, a bounded semilattice (hence a semigroup). $(S_E(U),\ \theta_\varepsilon)$ is a commutative monoid (hence a semigroup) whose identity is $\emptyset_\emptyset$ under the condition $T\cap Z\cap M=\emptyset$, where (F,T), (G,Z) and (H,M) are soft sets over U. Moreover, by Theorem 9 (i) (1), θ_ε distributes over $\cup_\varepsilon$ from LHS under $T\cap(Z\Delta M)=T\cap Z\cap M=\emptyset$, and, θ_ε distributes over $\cup_\varepsilon$ form RHS under the condition $(T\Delta Z)\cap M=T\cap Z\cap M=\emptyset$. Consequently, under the conditions $T\cap Z\cap M=(T\Delta Z)\cap M=T\cap(Z\Delta M)=\emptyset$, $(S_E(U),\ \cup_\varepsilon,\theta_\varepsilon)$ is an additive idempotent commutative semiring without zero, but with unity under certain conditions.

Corollary 3. $(S_E(U),\cap_\varepsilon,\theta_\varepsilon)$ is an additive idempotent commutative semiring without zero, but with unity under certain conditions.

Proof. Ali et al. (2011) showed that $(S_E(U),\cap_\varepsilon)$ is a commutative, idempotent monoid with identity $\emptyset_\emptyset$, that is, a bounded semilattice (hence a semigroup). $(S_E(U),\ \theta_\varepsilon)$ is a commutative monoid (hence a semigroup) whose identity is $\emptyset_\emptyset$ under the condition $T\cap Z\cap M=\emptyset$, where (F,T), (G,Z) and (H,M) are soft sets over U. Moreover, θ_ε distributes over $\cap_\varepsilon$ from LHS under $T\cap(Z\Delta M)=T\cap Z\cap M=\emptyset$, and θ_ε distributes over $\cap_\varepsilon$ from RHS under the condition $(T\Delta Z)\cap M=T\cap Z\cap M=\emptyset$. Consequently, under the condition $T\cap Z\cap M=(T\Delta Z)\cap M=T\cap(Z\Delta M)=\emptyset$. $(S_E(U),\cap_\varepsilon,\theta_\varepsilon)$ is an additive idempotent commutative semiring without zero, but with unity under certain conditions.

Theorem 10. Let (F,T), (G,Z), and (H,M) be soft sets over U. Then, extended theta operation distributes over soft binary piecewise operations as follows:

i) LHS Distributions

The following equations are satisfied if $T\cap Z\cap M=T\cap(Z\Delta M)=\emptyset$.

1)$(F,T)\ \theta_\varepsilon[(G,Z)\ \tilde{\cap}\ (H,M)]=[(F,T)\theta_\varepsilon(G,Z)]\ \tilde{\cap}[(F,T)\ \theta_\varepsilon\ (H,M)]$.

Proof. First, consider the LHS. Let $(G,Z)\ \tilde{\cap}\ (H,M)=(R,Z)$, where for every $\alpha\in Z$;

$$R(\alpha)=\begin{cases} G(\alpha), & \alpha \in Z-M \\ G(\alpha)\cap H(\alpha), & \alpha \in Z\cap M \end{cases}$$

$(F,T)\ \theta_\varepsilon(R,Z)=(N,T\cup Z)$, where for every $\alpha\in T\cup Z$;

$$N(\alpha)=\begin{cases} F(\alpha), & \alpha \in T-Z \\ R(\alpha), & \alpha \in Z-T \\ F'(\alpha)\cap R'(\alpha), & \alpha \in T\cap Z \end{cases}$$

Thus,

$$N(\alpha)=\begin{cases} F(\alpha), & \alpha \in T-Z \\ G(\alpha), & \alpha \in (Z-M)-T \\ G(\alpha)\cap H(\alpha), & \alpha \in (Z\cap M) \\ F'(\alpha)\cap G'(\alpha), & \alpha \in T\cap (Z\text{-}M) \\ F'(\alpha)\cap\ [G'(\alpha)\cup H'(\alpha)], & \alpha \in T\cap (Z\cap M) \end{cases}$$

Now consider the RHS, i.e. $[(F,T)\theta_\varepsilon(G,Z)]\ \tilde{\cap}[(F,T)\ \theta_\varepsilon\ (H,M)]$. Let $(F,T)\ \theta_\varepsilon(G,Z)=(K,T\cup Z)$, where for every $\alpha\in T\cup Z$;

$$K(\alpha)=\begin{cases} F(\alpha), & \alpha \in T-Z \\ G(\alpha), & \alpha \in Z-T \\ F'(\alpha)\cap R'(\alpha), & \alpha \in T\cap Z \end{cases}$$

Let $(F,T)\ \theta_\varepsilon(H,M)=(S,T\cup M)$, where for every $\alpha\in T\cup M$;

$$S(\alpha)=\begin{cases} F(\alpha), & \alpha \in T-M \\ H(\alpha), & \alpha \in M-T \\ F'(\alpha)\cap H'(\alpha), & \alpha \in T\cap M \end{cases}$$

Let $(K,T\cup Z)\ \tilde{\cap}(S,T\cup M)=(L,(T\cup Z)\cup(T\cup M))$, where for every $\alpha\in(T\cup Z)\cup(T\cup M)$;

$$L(\alpha)=\begin{cases} K(\alpha), & \alpha \in (T\cup Z)-(T\cup M) \\ K(\alpha)\cap S(\alpha), & \alpha \in (T\cup Z)\cap(T\cup M) \end{cases}$$

Thus,

$$L(\alpha)=\begin{cases} F(\alpha), & \alpha \in (T-Z)-(T\cup M)=\emptyset \\ G(\alpha), & \alpha \in (Z-T)-(T\cup M) \\ F'(\alpha)\cap G'(\alpha), & \alpha \in (T\cap Z)-(T\cup M)=\emptyset \\ F(\alpha)\cap F(\alpha), & \alpha \in (T-Z)\cap(T-M) \\ F(\alpha)\cap H(\alpha), & \alpha \in (T-Z)\cap(M-T)=\emptyset \\ F(\alpha)\cap\ [F'(\alpha)\cap H'(\alpha)], & \alpha \in (T-Z)\cap(T\cap M) \\ G(\alpha)\cap F(\alpha), & \alpha \in (Z-T)\cap(T-M)=\emptyset \\ G(\alpha)\cap H(\alpha), & \alpha \in (Z-T)\cap(M-T) \\ G(\alpha)\cap\ [F'(\alpha)\cap H'(\alpha)], & \alpha \in (Z-T)\cap(T\cap M)=\emptyset \\ [F'(\alpha)\cap G'(\alpha)]\cap F(\alpha) & \alpha \in (T\cap Z)\cap(T-M) \\ [F'(\alpha)\cap G'(\alpha)]\cap H(\alpha) & \alpha \in (T\cap Z)\cap(M-T)=\emptyset \\ [F'(\alpha)\cap G'(\alpha)]\cap[F'(\alpha)\cap H'(\alpha)], & \alpha \in T\cap Z\cap M \end{cases}$$

Thus,

$$L(\alpha)=\begin{cases} G(\alpha), & \alpha \in T\cap Z\cap M' \\ F(\alpha), & \alpha \in T\cap Z'\cap M' \\ \emptyset, & \alpha \in T'\cap Z'\cap M \\ G(\alpha)\cap H(\alpha), & \alpha \in T'\cap Z\cap M \\ \emptyset, & \alpha \in T\cap Z\cap M' \\ F'(\alpha)\cap\ [G'(\alpha)\cap H'(\alpha], & \alpha \in T\cap Z\cap M \end{cases}$$

When considering T-Z in the function N, since $T\text{-}Z=T\cap Z'$, if an element is in the complement of Z, it is either in M-Z, or $(M\cup Z)'$. Thus, if $\alpha\in T\text{-}Z$, then either $\alpha\in T\cap M\cap Z'$ or $\alpha\in T\cap M'\cap Z'$.Therefore, N=L under the condition $T\cap Z\cap M=T\cap Z'\cap M=T\cap Z\cap M'=\emptyset$. It is obvious that the condition $T'\cap Z\cap M=T\cap Z'\cap M=\emptyset$ is equivalent to the condition $(T\Delta Z)\cap M=\emptyset$.

2)$(F,T)\ \theta_\varepsilon\ [(G,Z)\ \tilde{\cup}\ H,M)]=[(F,T)\theta_\varepsilon(G,Z)]\ \tilde{\cup}\ [(F,M)\ \theta_\varepsilon\ (G,Z)]$.

ii) RHS Distributions

The following equations are satisfied if $(T\Delta Z)\cap M=T\cap Z\cap M=\emptyset$.

1)$[(F,T)\tilde{\cup}\ (G,Z)]\ \theta_\varepsilon\ (H,M)=[(F,T)\theta_\varepsilon(H,M)]\ \tilde{\cup}\ [(G,Z)\ \theta_\varepsilon(H,M)]$.

Proof. First, consider the LHS of the equality. Let $(F,T)\ \tilde{\cup}\ (G,Z)=(R,T)$, where for every $\alpha\in T$;

$$R(\alpha)=\begin{cases} F(\alpha), & \alpha \in T-Z \\ F(\alpha)\cup G(\alpha), & \alpha \in T\cap Z \end{cases}$$

Let $(R,T)\ \theta_\varepsilon\ (H,M)=(N,T\cup M)$, where for every $\alpha\in T\cup M$;

$$N(\alpha)=\begin{cases} R(\alpha), & \alpha \in T-M \\ H(\alpha), & \alpha \in M-T \\ R'(\alpha)\cap H'(\alpha), & \alpha \in T\cap M \end{cases}$$

Thus,

$$N(\alpha)=\begin{cases} F(\alpha), & \alpha \in (T-Z)-M \\ F(\alpha)\cup G(\alpha) & \alpha \in (T\cap Z)-M \\ H(\alpha), & \alpha \in M-T \\ F'(\alpha)\cap H'(\alpha), & \alpha \in (T-Z)\cap M \\ [F'(\alpha)\cap G'(\alpha)]\cap H'(\alpha), & \alpha \in T\cap(Z\cap M) \end{cases}$$

Now consider the RHS. Let $(F,T)\ \theta_\varepsilon(H,M)=(K,T\cup M)$, where for every $\alpha\in T\cup M$;

$$K(\alpha)=\begin{cases} F(\alpha), & \alpha \in T-M \\ H(\alpha), & \alpha \in M-T \\ F'(\alpha)\cap H'(\alpha), & \alpha \in T\cap M \end{cases}$$

Let $(G,Z)\ \theta_\varepsilon\ (H,M)=(S,T\cup M)$, where for every $\alpha\in Z\cup M$;

$$S(\alpha)=\begin{cases} G(\alpha), & \alpha \in Z-M \\ H(\alpha), & \alpha \in M-Z \\ G'(\alpha)\cap H'(\alpha), & \alpha \in Z\cap M \end{cases}$$

Let $(K,T\cup M)\ \tilde{\cup}\ (S,Z\cup M)=(L,(T\cup M)\cup(Z\cup M))$, where for every $\alpha\in(T\cup M)\cup(Z\cup M)$;

$$L(\alpha)=\begin{cases} K(\alpha), & \alpha \in (T\cup M)-(Z\cup M) \\ K(\alpha)\cup S(\alpha), & \alpha \in (T\cup M)\cap(Z\cup M) \end{cases}$$

$$L(\alpha)=\begin{cases} F(\alpha), & \alpha \in (T-M)-(Z\cup M) \\ H(\alpha), & \alpha \in (M-T)-(Z\cup M)=\emptyset \\ F'(\alpha)\cap H'(\alpha), & \alpha \in (T\cap Z)-(Z\cup M)=\emptyset \\ F(\alpha)\cup G(\alpha), & \alpha \in (T-M)\cap(Z-M) \\ F(\alpha)\cup H(\alpha), & \alpha \in (T-M)\cap(M-Z)=\emptyset \\ F(\alpha)\cup\ [G'(\alpha)\cap H'(\alpha)], & \alpha \in (T-M)\cap(Z\cap M)=\emptyset \\ H(\alpha)\cup G(\alpha), & \alpha \in (M-T)\cap(Z\text{-}M)=\emptyset \\ H(\alpha)\cup H(\alpha), & \alpha \in (M-T)\cap(M-Z) \\ H(\alpha)\cup\ [G'(\alpha)\cap H'(\alpha)], & \alpha \in (M-T)\cap(Z\cap M) \\ [F'(\alpha)\cap H'(\alpha)]\cup G(\alpha), & \alpha \in (T\cap M)\cap(Z\text{-}M)=\emptyset \\ [F'(\alpha)\cap H'(\alpha)]\cup H(\alpha), & \alpha \in (T\cap Z)\cap(M\text{-}Z)=\emptyset \\ [F'(\alpha)\cap H'(\alpha)]\cup[G'(\alpha)\cap H'(\alpha)], & \alpha \in T\cap Z\cap M \end{cases}$$

Hence,

$$L(\alpha)=\begin{cases} F(\alpha), & \alpha \in T \cap Z' \cap M' \\ F(\alpha) \cup G(\alpha), & \alpha \in T \cap Z \cap M' \\ H(\alpha), & \alpha \in T' \cap Z' \cap M \\ H(\alpha) \cup G'(\alpha), & \alpha \in T' \cap Z \cap M \\ F'(\alpha) \cup H(\alpha), & \alpha \in T \cap Z' \cap M \\ [F'(\alpha) \cup G'(\alpha)] \cap H'(\alpha), & \alpha \in T \cap Z \cap M \end{cases}$$

When considering M-T in the function N, since M-T=M∩T', if an element is in the complement of T, then it is either in Z-T or (Z∪T)'. Thus if α∈M-T, then α∈ M∩Z∩T' or α∈M∩Z'∩T'. Thus, N=L under T'∩Z∩M= T∩Z'∩M = T∩Z∩M =∅.

2) $[(F,T)\tilde{\cap}(G,Z)]\ \theta_\varepsilon\ (H,M) = [(F,T)\theta_\varepsilon(H,M)]\ \tilde{\cap}\ [(G,Z)\ \theta_\varepsilon\ (H,M)]$.

Corollary 4. $(S_E(U),\tilde{\cup},\theta_\varepsilon)$ is an additive idempotent multiplicative commutative semiring without zero, but with unity under certain conditions.

Proof. Yavuz (2024) showed that $(S_E(U),\tilde{\cup})$ is an idempotent, non-commutative semigroup (that is a band) under the condition T∩Z'∩M =∅, where (F,T), (G,Z) and (H,M) are soft sets. $(S_E(U),\theta_\varepsilon)$ is a commutative monoid (hence a semigroup) whose identity is $\emptyset_\emptyset$ under the condition T∩Z∩M=∅, where (F,T), (G,Z) and (H,M) are soft sets over U. Moreover, θ_ε distributes over $\tilde{\cup}$ from LHS under T∩Z∩M=T∩(ZΔM)=∅, and θ_ε distributes over $\tilde{\cup}$ from RHS under the condition (TΔZ)∩M=T∩Z∩M =∅. Consequently, under the conditions T∩Z∩M=T∩(ZΔM)=(TΔZ)∩M=T∩Z'∩M=∅, $(S_E(U),\tilde{\cup},\theta_\varepsilon)$ is an additive idempotent multiplicative commutative semiring without zero, but with unity under certain conditions.

Corollary 5. $(S_E(U),\tilde{\cap},\theta_\varepsilon)$ is an additive idempotent multiplicative commutative semiring without zero, but with unity under certain conditions.

Proof. Yavuz (2024) showed that $(S_E(U),\tilde{\cap})$ is an idempotent, commutative semigroup (that is a band) under the condition T∩Z'∩M =∅, where (F,T), (G,Z) and (H,M) are soft sets. $(S_E(U),\theta_\varepsilon)$ is a commutative monoid (hence a semigroup) whose identity is $\emptyset_\emptyset$ under the condition T∩Z∩M=∅, where (F,T), (G,Z) and (H,M) are soft sets over U. Moreover, θ_ε distributes over $\tilde{\cap}$ from LHS under T∩Z∩M=T∩(ZΔM)=∅ and θ_ε distributes over $\tilde{\cap}$ from RHS under the condition (TΔZ)∩M=T∩Z∩M=∅. Consequently, under the conditions T∩Z∩M=T∩(ZΔM)=(TΔZ)∩M=T∩Z'∩M=∅, $(S_E(U),\tilde{\cap},\theta_\varepsilon)$ is an additive idempotent multiplicative commutative semiring without zero, but with unity under certain conditions.

CONCLUSION

Parametric techniques like soft sets and soft operations are very useful when dealing with uncertainty. Introducing new soft operations and figuring out their algebraic properties and uses opens up new ways to solve problems with parametric data. This work introduces a novel restricted and extended soft set operation in this manner. By putting out the idea of "restricted and extended theta operations of soft sets" and by carefully examining the algebraic structures associated with these and other specific kinds of soft set operations, we hope to make a meaningful contribution to the field of soft set theory. Specifically, an extensive analysis is conducted on the algebraic characteristics of these new soft set operations. Taking into account the algebraic properties of these soft set operations and distribution laws, a thorough study of the algebraic structures formed by these operations in the collection of soft sets over the universe is presented. We demonstrate that, under some assumptions, $(S_E(U),\theta_\varepsilon)$ is a commutative monoid with identity $\emptyset_\emptyset$. Furthermore, we demonstrate how several significant algebraic structures, including semirings, are formed in the collection of soft sets over the universe combined with extended theta operations and other kinds of soft set operations: $(S_E(U),\cup_\varepsilon,\theta_\varepsilon)$, $(S_E(U),\cap_\varepsilon,\theta_\varepsilon)$ are all additive idempotent commutative semirings without zero but with unity under certain conditions. $(S_E(U),\tilde{\cap},\theta_\varepsilon)$, $(S_E(U),\tilde{\cup},\theta_\varepsilon)$ are all additive idempotent multiplicative commutative semirings without zero but with unity under certain conditions.

By examining novel soft set operations and the algebraic structures of soft sets, we can fully comprehend their application. This can advance soft set theory and the classic algebraic literature in addition to offering new examples of algebraic structures. Future research might look at other varieties of new restricted and extended soft set operations, as well as the matching distributions and characteristics, to add to this body of knowledge.

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